Iterative Design of IIR Variable Fractional Delay Digital Filters

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Abstract—In this paper, an iterative algorithm is proposed to design IIR variable fraction delay (VFD) digital filters in the weighted least-squares (WLS) sense. The original IIR VFD filter design problem is nonconvex. As an attempt to tackle this difficulty, an iterative procedure is introduced, and the Steiglitz-McBride (SM) reweighting technique is employed at each iteration to transform the original approximation error into a (convex) quadratic form. The stability of designed IIR VFD filters can be ensured by imposing a set of linear stability constraints based on the positive realness. The proposed algorithm can be applied to design IIR VFD filters with either fixed or variable denominator. Two examples are presented to illustrate the effectiveness of the proposed algorithm.

I. INTRODUCTION

Variable fractional delay (VFD) digital filters have been widely applied in signal processing and communications [1], such as sound synthesis, sampling rate conversion, and digital modem synchronization. So far, various methods for finite impulse response (FIR) and allpass VFD filter design [1]-[8] have been advanced. In general, optimal FIR VFD filter design can be obtained by numerical procedures [1]-[2] or in closed forms [4]-[5]. Compared with FIR VFD filter design, allpass VFD filter design faces additional challenge due to the presence of the denominator. Since allpass VFD filters have fullband unity magnitude responses, some algorithms [2], [6]-[7] have been formulated to minimize the approximation error of group delay responses. Using the mirror symmetric relation between the numerator and the denominator of an allpass VFD filter, some other algorithms [1], [8] minimize the approximation error in terms of frequency responses of the denominator.

General IIR digital filters have recently been advanced for VFD digital filter design [1], [9]-[15]. The results in [1] show that general IIR digital filters exhibit low delay and wide-band capabilities in VFD filter design. The IIR VFD filter design algorithms [1], [9]-[15] can be generally classified as two-stage approach and integrated approach. Under the two-stage approach [1], [9]-[14], a set of stable IIR digital filters with sampled fractional delays (FD) are designed first, and then the polynomial coefficients are determined by fitting the obtained IIR FD filter coefficients in the least-squares (LS) sense. Under the integrated approach [15], optimization is carried out directly on the polynomial coefficients of the numerator. In [9]-[13], both the numerator and denominator coefficients are variable. In [1], [9], [14], [15], only the numerator coefficients are variable.

In this paper, an iterative algorithm is to be proposed for designing IIR VFD filters with either fixed or variable denominator. The remaining part of this paper is organized as follows: The iterative design algorithm is to be described in Section II. Two numerical examples are to be presented in Section III. Conclusions are to be made in Section IV.

II. PROPOSED ITERATIVE DESIGN ALGORITHM

A. Problem Formulation

Let the ideal frequency response of a VFD digital filter be defined as

\[ H_d(\omega, d) = e^{-j\alpha D + j\omega d}, \quad \omega \in [0, \pi\alpha] \]  

where \( 0 < \alpha < 1, \) \( D \) is an integer delay, and \( d \) is a variable fractional delay within the range \([-0.5, 0.5]\). The transfer function of an IIR VFD filter can be expressed as

\[
H(z, d) = \frac{P(z, d)}{Q(z, d)} = \frac{\sum_{n=0}^{N} p_n(d) z^{-n}}{1 + \sum_{m=1}^{M} q_m(d) z^{-m}} = \frac{\varphi^r(z) p(d)}{1 + \varphi^l(z) q(d)}
\]

where

\[
\varphi_r(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-N} \end{bmatrix}^T
\]

\[
\varphi_l(z) = \begin{bmatrix} z^{-1} & z^{-2} & \cdots & z^{-M} \end{bmatrix}^T
\]

\[
p(d) = [p_0(d), p_1(d), \ldots, p_N(d)]^T
\]

\[
q(d) = [q_1(d), q_2(d), \ldots, q_M(d)]^T
\]
The superscript $T$ used in (2)-(6) denotes the transposition of a vector or matrix. All the filter coefficients $p_n(d)$ and $q_{n}(d)$ can be expanded as polynomial functions of the fractional delay $d$

$$p_n(d) = \sum_{k=0}^{K_1} a_{nk} d^k = [a_n]^T d$$

$$q_n(d) = \sum_{k=0}^{K_1} b_{nk} d^k = [b_n]^T d$$

where

$$v_1(d) = [1 \ d \ \cdots \ d^{K_1}]^T$$

$$v_2(d) = [1 \ d \ \cdots \ d^{K_1}]^T$$

$$a_n = [a_{n0} \ a_{n1} \ \cdots \ a_{nK_1}]^T$$

$$b_n = [b_{n0} \ b_{n1} \ \cdots \ b_{nK_1}]^T$$

As $K_1 = 0$, the designed IIR VFD filter has a fixed denominator. In this case, $Q(z, d)$ and $Q(d)$ can be simply written as $Q(z)$ and $q$. For convenience, we can define a numerator coefficient vector $a$ as

$$a = [a_0^T \ a_1^T \ \cdots \ a_{n-1}^T]^T$$

And the denominator coefficient vector $b$ can be defined as

$$b = [b_0^T \ b_1^T \ \cdots \ b_{n-1}^T]^T$$

Then, $P(z, d)$ and $Q(z, d)$ can be simplified as

$$P(z, d) = a^T u_z(z, d)$$

$$Q(z, d) = 1 + b^T u_z(z, d)$$

where

$$u_z(z, d) = [z^{-1} v_1^T(d) \ \cdots \ z^{-n} v_1^T(d)]$$

$$u_z(z, d) = [z^{-1} v_2^T(d) \ \cdots \ z^{-n} v_2^T(d)]$$

In the WL S sense, the design problem can be expressed as

$$\min_{\omega} J(x) = \frac{1}{\mu} \int_{-\alpha \pi}^{\alpha \pi} W(\omega, z) |x(\omega, z)|^2 d\omega$$

where $x = [a^T \ b^T]^T$, $W(\omega, z)$ is the specified nonnegative weighting function, and the complex approximation error $e(\omega, z)$ is defined as

$$e(\omega, z) = H_1(e^{j\omega} z) - H_2(\omega, d)$$

B. Proposed Iterative Design Algorithm

Note that the design problem (19) is highly nonconvex. The Steiglitz-McBride reweighting technique [16] is adopted here in which an iterative procedure is employed by the proposed design algorithm. At the $i$th iteration, the integrand of the objective function in (19) can be rewritten as

$$W(t^{(i)}(\omega, d)) = \left| P(t^{(i)}(e^{j\omega} d)) - H_2(\omega, d) Q(t^{(i)}(e^{j\omega} d)) \right|^2$$

$$= W(t^{(0)}(\omega, d)) x^{(i)}(\omega, d) \Re \{ U(\omega, d) \} x^{(i)}(\omega, d)$$

$$- 2 W(t^{(0)}(\omega, d)) x^{(i)}(\omega, d) \Re \{ u(\omega, d) e^{j(D+4\pi)} \}$$

$$+ W(t^{(0)}(\omega, d))$$

where

$$W^{(i-1)}(\omega, d) = \frac{1}{\left| Q(\omega, d) \right|^2}$$

$$u(\omega, d) = \left| u(t^{(i)}(e^{j\omega} d)) - e^{-j(D+4\pi)\omega} u(t^{(i)}(e^{j\omega} d)) \right|^2$$

$$U(\omega, d) = u(\omega, d) u^{(i)}(\omega, d)$$

In (21), $\Re \{ \cdot \}$ denotes the real part of a complex variable. The superscript $H$ in (24) represents the transpose-conjugate of a vector or matrix. By using $W^{(i)}(\omega, d)$ in (21), the objective function of (19) can be expressed as

$$J(x^{(i)}) = x^{(i)T} G^{(i)} x^{(i)} - 2 x^{(i)T} b^{(i)} + \text{constant}$$

where

$$G^{(i)} = \int_{-\alpha \pi}^{\alpha \pi} W(t^{(i)}(\omega, z)) \Re \{ U(\omega, z) \} d\omega$$

$$b^{(i)} = \int_{-\alpha \pi}^{\alpha \pi} W(t^{(i)}(\omega, z)) \Re \{ u(\omega, z) e^{j(D+4\pi)\omega} \} d\omega$$

With the stability constraint to be described in the next subsection, the design problem can be formulated as a quadratic programming (QP) problem.

In order to improve the robustness of the proposed iterative procedure described above, the following iterative scheme is further employed to update the filter coefficients $x^{(i)}$.

$$x^{(i)} = x^{(i)} + (\lambda - 1) x^{(i-1)}$$

In (28), $0 < \lambda < 1$ is a relaxation constant, and $\Psi$ is the operator mapping $x^{(i)}$ to the solution of (28). The iterative procedure continues until the following condition is satisfied

$$\frac{J(x^{(i)}) - J(x^{(i-1)})}{J(x^{(i)})} \leq \mu$$

where $\mu$ is a small tolerance specified by designers. The convergence of the iterative procedure can be ensured since the WLS error cannot be infinitely reduced.

C. Stability Constraint

In order to guarantee the stability of designed IIR digital filters, stability constraints have to be incorporated in the iterative design procedure. For the sake of discussion, the stability constraint is to be introduced first for the case of designing IIR VFD filters with fixed denominator. Then, it can be readily extended to the situation of designing IIR VFD filters with variable denominator.

A sufficient condition for the stability of IIR digital filters has been presented in [17], which states that if all the roots of $Q^{(i)}(z)$ lie inside the unit circle and the transfer function $R(t^{(i)}(z)) = Q^{(i)}(z)/Q^{(i)}(z)$ is strictly positive real on the unit circle, all the roots of the polynomial $Q^{(i)}(z) = (1-\lambda)Q^{(i)}(z)+\alpha Q^{(i)}(z)$ for $\forall \lambda \in [0, 1]$ lie inside the unit circle. Note that the requirement that $\Re \{ R(t^{(i)}(e^{j\omega})) \} > 0$ for $\forall \omega \in [0, \pi]$ is equivalent to $R^{(0)}(e^{j\omega}) + R^{(0)}(e^{-j\omega}) > 0$. In [17], by introducing auxiliary variables, this stability condition can be further cast as a linear matrix inequality (LMI) constraint. For simplicity, we shall express the stability constraint $R^{(0)}(e^{j\omega}) + R^{(0)}(e^{-j\omega}) > 0$ as

$$\Re \{ G^{(i)}(e^{j\omega}) \} \geq 0$$

$$\omega \in [0, \pi], \ i = 1, \cdots, I$$

164
where \( v \) is a specified small positive number.

As a variable denominator is utilized in (2), the above stability constraint is transformed as

\[
\text{Re}\left\{G^{(i-1)}(e^{-j\omega}, d_i)G_{ii}^{(i)}(e^{j\omega}, d_i)\right\}b^{(i)} \geq v - \text{Re}\left\{G^{(i-1)}(e^{j\omega}, d_i)\right\}
\]

(31)

\( \omega_i \in [0, \pi], \ i = 1, \ldots, J \)
\( d_j \in [-0.5, 0.5], \ j = 1, \ldots, J \)

The stability constraints (30) and (31) can be readily extended to the situation where all the poles of obtained IIR VFD filters are required to lie inside a circle of radius \( \rho < 1 \).

### III. Simulations

In this section, two numerical examples are to be presented to demonstrate the effectiveness of the proposed design algorithm. In order to evaluate the performances of designed IIR VFD filters, the maximum absolute errors (MAE) \( e_{\text{max}} \) and the normalized root-mean-squared errors (RMSE) \( e_{\text{rms}} \) of frequency responses are adopted and defined as

\[
e_{\text{max}} = \max \{ |e(\omega, d)|, \ \omega \in [0, \alpha] \pi, d \in [-0.5, 0.5] \}
\]

(32)

\[
e_{\text{rms}} = \left[ \frac{1}{\alpha \pi} \int_{0}^{\alpha \pi} |e(\omega, d)|^2 d\omega \right]^{1/2}
\]

(33)

In both examples, the initial point is always chosen as \( x^{(0)} = 0 \), where \( \theta \) is a zero vector. The weighting function \( W(\omega, d) \) is set equal to 1 over the domain \( [0, \alpha \pi] \times [-0.5, 0.5] \). The parameters \( \lambda, \mu, \) and \( v \) are chosen, respectively, as 0.5, 10\(^4\), and 10\(^{-3}\). The stability constraints (31) are imposed on \( 21 \times 21 \) discrete points evenly distributed over the domain \( [0, \pi] \times [-0.5, 0.5] \), while \( K_1 = 0 \), the stability constraints (30) are imposed on 21 frequency points equally spaced on the range \( [0, \pi] \). At each iteration, the QP design problem is solved by the command "quadprog" in MATLAB.

The first example is to design an IIR VFD filter with a fixed denominator, i.e., \( K_2 = 0 \). The filter orders are chosen as \( N = 30 \) and \( M = 10 \). The integer delay \( D \) is set equal to 16. The polynomial order of numerator is set as \( K_3 = 6 \). The cutoff frequency is 0.9\( \pi \), i.e., \( \alpha = 0.9 \). After 20 iterations, the design procedure converged to the final solution. The maximum pole radius of the obtained IIR VFD filter is 0.922. The magnitude and group delay responses of the obtained IIR VFD filter are shown in Fig. 1. The magnitude of the complex approximation error \( e(\omega, d) \) is given in Fig. 2. For comparison, we also utilized the SOCP algorithm [15] to design an IIR VFD filter under the same set of specifications. The fixed denominator was designed by the model-reduction method [18] from an average FIR VFD filter, which was obtained by the WLS algorithm [5] with the filter order 61 and the polynomial order 6. The maximum pole radius of the IIR VFD filter obtained by the SOCP design method [15] is 0.908. All the error measurements are summarized in Table 1. It can be observed that the proposed algorithm can achieve better performance than the SOCP algorithm [15].

| Table I. Measurements of IIR VFD Filters in Example 1 |

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>MAE ( e_{\text{max}} ) (in dB)</th>
<th>RMSE ( e_{\text{rms}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>-63.784</td>
<td>1.006e-4</td>
</tr>
<tr>
<td>SOCP [15]</td>
<td>-56.808</td>
<td>1.359e-4</td>
</tr>
</tbody>
</table>

In the second example, an IIR VFD filter with a variable denominator is to be designed using the proposed method. The filter orders are chosen as \( N = 40 \) and \( M = 10 \). The integer delay \( D \) is set as 21. The polynomial orders are \( K_1 = K_3 = 5 \). The cutoff frequency is also set equal to 0.9\( \pi \). After 17 iterations, the iterative procedure stopped at the final solution. It can be observed that the obtained IIR VFD filter is always stable. The magnitude and group delay responses of the obtained IIR VFD filter are shown in Fig. 3. The magnitude of the complex approximation error \( e(\omega, d) \) is given in Fig. 4. The maximum pole radius versus the fractional delay \( d \) is illustrated in Fig. 5. For comparison, we also designed an FIR VFD filter using the WLS design algorithm [5] under the same set of specifications except the FIR filter order chosen as \( N+M = 50 \). It should be noticed that the obtained FIR VFD filter by the WLS algorithm [5] is globally optimal. All the error measurements are summarized in Table II. Obviously, by using the same number of tunable filter coefficients, the obtained IIR VFD filter can achieve much better performance than the FIR VFD filter.
IV. CONCLUSIONS

In this paper, an iterative algorithm for designing IIR VFD filters in the WLS sense has been proposed. To simplify the original design problem, an iterative procedure is introduced. At each iteration, the SM reweighting technique is utilized and accordingly the objective function is cast as a (convex) quadratic form. In order to ensure the stability of designed IIR VFD filters, a set of linear inequality constraints derived from a sufficient condition in terms of the positive realness are incorporated. The proposed algorithm can be applied to design IIR VFD filters with either fixed or variable denominator. Two filter examples have been presented which indicate that the proposed design algorithm can achieve better results than those of two existing methods.

REFERENCES


