Abstract—The Farrow structure provides an effective means of implementing a variable fractional delay filter. In the literature, the design of Farrow filters is invariably formulated as a weighted least-squares problem, or similar. However, this design approach does not often yield satisfactory group-delay responses. This paper presents a design formulation that addresses this shortcoming directly. It also explores a more generalised form of the Laguerre-Farrow filter.

I. INTRODUCTION

Fractional delay (FD) filters have found applications in many signal processing fields, including communication, speech processing, and array processing. These filters allow the user to delay sampled signals by amounts that are not integer multiples of the sampling period. An excellent discussion on the various design methods for fractionally delay filters can be found in the tutorial article [1].

In many advanced applications there are demands for the delay to be continuously variable during the operation of the filter. These filters are known as variable fractional delay (VFD) filters. VFD filters are very useful in applications such as time-delay estimation, speech coding, signal interpolation, and sampling rate conversion. Published works include variable filter banks for audio signal processing [2], adaptive noise reduction [3], reduction of channel timing-error effects in time-interleaved AD converters [4], modelling of music instruments, speech coding and others where new samples need to be interpolated [1], wireless radio optimisation [5], multi-mode transmultiplexers [6], fractional delay Hilbert transform filters [7], and IIR variable delay filters [8]. All of the above-mentioned publication had one thing in common and that is the use of the Farrow structure [9] as the foundation of their designs.

In this paper, the discussion will include a reformulation of the optimum design problem to allow frequency response shaping. It also shows the generalization of the conventional Farrow structure to incorporate Laguerre filters [8]. Laguerre filters are higher order forms of the unit delay elements of an FIR filter. By replacing the unit delays with Laguerre filter sections, extra degrees of freedom are introduced into the design. It will be shown that this extra freedom can yield a better design.

II. THE LAGUERRE-FARROW STRUCTURE

A. The Conventional Farrow Structure

The celebrated Farrow structure is shown in Fig. 1. It consists of M parallel FIR filters, whose coefficients are fixed, and a multiplier chain through which the user adjusts the filter delay δ. It can be readily shown from Fig. 1 that the Farrow filter has transfer function

\[ H(z, \delta) = \sum_{k=0}^{M-1} \sum_{m=0}^{K-1} b_{k,m} \delta^m z^{-k} \]  

(1)

One view of the Farrow structure is that it is simply a K-tap FIR filter whose coefficient are each an (M-1)th order polynomial in \( \delta \). In other words, with an appropriate choice of filter coefficients \( b_{k,m} \in \mathbb{R}, \quad k = 0, 1, \ldots, (K - 1), \quad m = 0, 1, \ldots, (M - 1), \) the Farrow filter aims to approximate, as closely as possible, over a range of delays \( \delta \), the ideal variable fractional delay filter whose frequency response is given by

\[ H_f(e^{j\omega}, D, \delta) = e^{-jD(\delta)} \]  

(2)

where \( D \) is the nominal, or bulk, delay of the filter. Methods to design the M fixed FIR filters are summarized in [1].
B. Laguerre-Farrow Formulation

In [10] Cook et al. investigated the Farrow structure and proposed a generalised structure called the Laguerre-Farrow structure. The Cook proposal can be generalised and formulate as follows,

\[ H(z, \delta) = \sum_{k=0}^{M-1} \sum_{m=0}^{K-1} h_{k,m} \varphi_{k,m}(z) \delta^m \]  

(3)

where

\[ \varphi_{k,m}(z) = \sqrt{1 - \alpha_{k,m}^2} \frac{(z^{-1} - \alpha_{k,m})^\ell}{1 - \alpha_{k,m} z^{-1}} \]  

(4)

and \(|\alpha_{k,m}| < 1, \alpha_{k,m} \in \mathbb{R}\). Note that in (3), \(\varphi_{k,m}(z)\) forms an orthonormal set of basis function [11], and if \(\varphi_{k,m}(z)\) is as defined by (4), these function are called Laguerre function. Also note that a Laguerre-Farrow filter reduces to an FIR filter when \(\alpha_{k,m} = 0\). The new proposed filter structure is shown in Fig. 2 where

\[ L_{k,m}(z) = \frac{z^{-1} - \alpha_{k,m}}{1 - \alpha_{k,m} z^{-1}} \]  

(5)

and

\[ T_{k,m}(z) = \sqrt{1 - \alpha_{k,m}^2} \frac{1}{1 - \alpha_{k,m} z^{-1}} \]  

(6)

Evidently, the extra poles give more degrees of freedom to improve the design.

\[ x(n) \quad T_{k,m} \quad h_{k,m} \quad y(n) \]

Observing Fig. 2 one may argue that the Laguerre-Farrow structure is significantly more complex to implement than the Farrow structure of Fig. 1. Whilst the Laguerre-Farrow structure is undeniably more complex, there are structural simplifications that one can exploit to reduce its complexity. In [10], Cook et al. demonstrated one form of simplification which we called the 1-pole Laguerre-Farrow fractional delay filter. This simplification requires the poles of the Laguerre-Farrow structure to be equal. In [12], Burton and Leung further expanded the simplification to a 1-pole-per-row Laguerre-Farrow design. In both cases an improvement from the original Farrow design is clearly outlined. In this paper a more superior design is proposed as shown in Fig. 2.

C. Optimum Design Formulations

The authors in [10] used weighted least square (WLS) for their optimum design formulations. A possible alternative for the optimum design method is the minimax technique which has been explored in [13]. Upon further investigation one finds that additional improvement can be made by decoupling (3) into phase and magnitude response. Combining the two ideas, the following weighted minimax (WMM) problem is proposed: Minimising the phase error subject to amplitude masking. In the other words, a flatter phase delay response can be achieved with a little amplitude response trade-off. Mathematically the proposed optimisation problem can be expressed as follows.

\[ \min_{\alpha \in [0,0.5,1,1.5,...,\lambda]} \mathcal{P}(D) \]  

(7)

where

\[ \mathcal{P}(D) = \min_{\{a_m \in \mathbb{R}\}_D} \max_{0 \leq \omega \leq \omega_{\text{max}}} \left| \angle H_e \left( e^{j\omega}, D, \delta \right) \right| \]  

(8)

subject to

\[ |\alpha| < 1, \quad m = 0,...,(M-1) \]  

(9)

\[ D = 0.5 i, \quad i \in \mathbb{Z}^+ \]  

(10)

and

\[ W \left( e^{j\omega} \right) |H_e| < \varepsilon, \quad 0 \leq \omega \leq \pi \]  

(11)

\[ -0.5 < \delta < 0.5 \]  

(12)

where

\[ \left| H_e \right| = \left| H \left( e^{j\omega}, D, \delta \right) \right| - \left| H_d \left( e^{j\omega}, D, \delta \right) \right| \]  

(13)

and \(W(e^{j\omega}) \in \mathbb{R}^+\) is a weighting function, \(\omega_{\text{min}} \leq \omega \leq \omega_{\text{max}}\) is the range of frequency over which we require the variable delay filter to exhibit linear phase, and the phase response of the idea variable fractional delay filter is given by

\[ \angle H_d \left( e^{j\omega}, D, \delta \right) = -(D + \delta) \omega \]  

(14)

The purpose for the constraint (10) is to guarantee stable Laguerre filters, while (11) limits the choice of \(D\) to integer multiples of 0.5. The reason for this restriction on \(D\) stems from practical considerations in the application of variable fractional delay filters. The trade-off between the magnitude and phase responses of the design filter is controlled by \(\varepsilon\) (12). A larger \(\varepsilon\) allows for greater distortion in the magnitude response which in turn should result in less distortion in the phase response.

III. DESIGN EXAMPLES

A. Introduction

In this section, the designs of Laguerre-Farrow filters are presented and compared, namely, 1-pole design and row-
column design, along with the conventional Farrow filter using the weighted minimax constraint. The bases of our comparison are the partial all-pass and the band-pass response. Our objective is to analyse the maximum phase error for each of the above listed structures.

For each tested design the filter length is set constant to \( K = 8 \) and the delay variation of 1 sampling period, i.e. \( \delta = 0.5 \). Concerning \( M \), it was found experimentally that for \( \delta = 0.5 \), increasing \( M \) beyond 5 will give only a very small decrease in optimum cost. Thus we set \( M = 5 \). It was also found experimentally that \( M \) depends only very slightly on \( K \), which is expected since \( M \) gives the degree of the polynomial approximation of the filter coefficients on \( \delta \) as \( \delta \) varies from \( \delta - \delta \) to \( \delta \). The magnitude error bound is set to \( \epsilon = 0.1 \).

We evaluated \( H \left( e^{j \omega \delta}, \delta \right) \), and \( H_s \left( e^{j \omega \delta}, \delta \right) \) at 500 equally spaced frequency points from 0 to \( \pi \), and 10 equally spaced delay values from -0.5 to 0.5.

The simulated results were achieved through the use of MATLAB and its respective optimization toolbox.

### B. Partial All-Pass Design

The partial all-pass (AP) is so termed because the response is designed to exhibit a passband characteristic from \( \omega = 0 \) to \( 0.8\pi \) and a “don’t care” response from \( 0.8\pi \) to \( \pi \). From (8), we assigned \( \omega_m = 0 \), \( \omega_p = 0.8\pi \), and no weighting is applied.

The maximum phase errors of the WMM Partial AP comparison design are summarized in TABLE I. Maximum phase error improvements can be seen from Farrow to Laguerre-Farrow row-column structure; the maximum phase error improvement is thirty-folds above the conventional Farrow.

### TABLE I

<table>
<thead>
<tr>
<th>Structure</th>
<th>Max Phase Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Farrow</td>
<td>0.005803633</td>
</tr>
<tr>
<td>1-Pole</td>
<td>0.001829263</td>
</tr>
<tr>
<td>Row-Column</td>
<td>0.000174808</td>
</tr>
</tbody>
</table>

The magnitude response of the Farrow structure with weighted minimax partial AP filter is shown in Fig. 3. From 0 to \( 0.8\pi \) each delay response are well enclosed within the design constraint of \( \epsilon = 0.1 \) with maximum magnitude error being \(-7.5196 \times 10^{-3} \). The magnitude responses for the two Laguerre-Farrow designs are omitted due to the similar characteristics displayed in the graphs.

The design for the phase delay graphs uses the optimum bulk delay of \( D = 2.5 \). The ideal phase delay response is to have parallel horizontal line across the desired region of interest and its respective fractional delay steps. From Fig. 4 and Fig. 5 one can clearly see the significant phase improvements over the range of interest.

### C. Band-Pass Design

The design specifications for the band-pass (BP) design used in this design example are stopband \([0, 0.125\pi] \cup [0.875\pi, \pi]\), and passband \([0.375\pi, 0.625\pi]\). Optimum
bulk delay for this band-pass design is $D = 2.5$ and the tolerance constraint $\varepsilon = 0.1$. The weighting function in (12) is declared as

$$W(e^{j\omega}) = \begin{cases} 1, & 0 \leq \omega \leq 0.125\pi, \ 0.875\pi < \omega \leq \pi \\ 0, & 0.375\pi < \omega \leq 0.625\pi \end{cases}$$

The maximum phase error of WMM BP filters are summarised in TABLE II. In contrast to the Farrow structure, the Laguerre-Farrow structures show a significant improvement in the maximum phase error.

**TABLE II**

<table>
<thead>
<tr>
<th>Structure</th>
<th>Max Phase Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Farrow</td>
<td>0.018775987</td>
</tr>
<tr>
<td>1-Pole</td>
<td>0.003288857</td>
</tr>
<tr>
<td>Row-Column</td>
<td>0.002250036</td>
</tr>
</tbody>
</table>

The phase delay response graphs are shown in Fig. 7, Fig. 8 and Fig. 9. A progressive improvement in phase delay response can be seen from the Farrow structure to the row-column Laguerre-Farrow structure. This shows that Laguerre-Farrow in general has a better phase response and also a more accurate fractional phase approximation.

**IV. CONCLUSION**

In this paper, we have generalized the Farrow variable fractional delay structure to include Laguerre filters, and using a simple minimax design formulation, we have shown that the Laguerre-Farrow structure can offer performance gains. It is expected that by a more appropriate formulation, one can extract better performance gains from the higher order variable fractional delay filters. We also showed how one can include amplitude response shaping into the design formation and extend the range of delays that the filters can synthesize.
REFERENCES


