Low-Order Fixed Denominator IIR VFD Filter Design

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Abstract—A two-stage design method of low-order fixed denominator IIR variable fractional delay (VFD) digital filters is presented in this paper. In the first stage, a set of FIR fractional delay (FD) filters are designed first. Each FIR FD filter design problem is formulated in the peak-constrained weighted least-squares (PCWLS) sense and solved by the projected least-squares (PLS) algorithm. Then, model reduction technique is applied on a time-domain average FIR filter to obtain the fixed denominator. The remaining numerators of the IIR FD filters can be obtained by solving linear equations derived from the orthogonality principle. In the second stage of the design, these FD filter coefficients are to be approximated by polynomial functions of FD. Three sets of filter-examples are given to illustrate the effectiveness of the proposed design method.

I. INTRODUCTION

Variable fractional delay (VFD) digital filters are widely used in various signal processing applications [1]. The ideal fractional delay digital filter requires fullband constant group delay (or linear phase) responses achievable by linear-phase FIR digital filters and fullband unity magnitude responses achievable by allpass digital filters. The study of a general IIR digital filter as a VFD filter [2]-[8] was motivated by the hypothesis (made by H. K. Kwan) that a general IIR digital filter, which can offer a property that exhibits an overall combined reduced constant group delay and unity magnitude to a greater extent than that offered individually by an FIR digital filter or an allpass digital filter, each with the same number of distinct coefficients. Compared with FIR and allpass VFD filter designs, the design of an IIR VFD filter faces more challenges, since both its magnitude and group delay responses have to be approximated. However, an IIR VFD filter can achieve a lower group delay within frequency regions of interest. Recently, a number of methods [2]-[8] have been developed to design IIR VFD filters. By expressing each filter coefficient as a polynomial function of fractional delay (FD), a VFD filter can be realized using the Farrow’s structure [9]. In [2]-[6], both the denominator and numerator coefficients of a designed IIR VFD filter are variable. In [6]-[8], fixed denominator coefficients are considered. In this paper, IIR VFD filters are to be designed using low-order fixed denominators.

The paper is organized as follows: In Section II, the design problem is formulated. The two-stage design method is presented in Section III. Three sets of filter-examples are designed and analyzed in Section IV. Finally, conclusions are given in Section V.

II. PROBLEM FORMULATION

The ideal frequency response of a VFD digital filter is defined as

$$H_j(e^{j\omega}, d) = e^{-j(D+d)\omega} \quad \omega \in [0, w_c]$$

(1)

where $D$ is a positive integer delay, $d$ is a variable fractional delay in the range of $[-0.5, 0.5]$, and $w_c \in [0, \pi]$ is the cutoff frequency. The transfer function of an IIR VFD filter with a fixed denominator can be expressed as

$$H(z, d) = \frac{P(z, d)}{Q(z)} = \frac{\sum_{n=0}^{N} p_n(d) z^{-n}}{1 + \sum_{m=1}^{M} q_m z^{-m}} = p(d)^T \varphi_0(z)$$

(2)

where $p(d) = [p_0(d), \ldots, p_N(d)]^T$, $q = [1, q_1, \ldots, q_M]^T$, and $\varphi_0(z) = [1, z^1, \ldots, z^N]^T$. The superscript $T$ in (2) denotes the transpose operation of a vector. All the filter coefficients are assumed to be real values. Each numerator coefficients $p_n(d)$ is expressed as a polynomial function of the fractional delay $d$

$$p_n(d) = \sum_{k=0}^{K} a_n(k) d^k \quad n = 0, \ldots, N$$

(3)

The design task is to find a stable IIR VFD filter whose frequency response can best approximate the ideal frequency response $H_0(e^{j\omega}, d)$. For simplicity, the fractional delay $d$ is assumed to be uniformly sampled within $[-0.5, 0.5]$, i.e.,

$$d_l = -0.5 + \frac{l}{L} \quad l = 0, \ldots, L$$

(4)

III. IIR VFD DIGITAL FILTER DESIGN

In this paper, we propose a two-stage design method. In the first stage, a set of IIR FD digital filters are to be designed using the model reduction technique [10]. In the second stage, the numerator coefficients $p_n(d)$ are to be approximated by a polynomial function defined by (3).
A. FIR FD Filter Design

Before applying the model reduction method to design IIR FD filters, a set of FIR FD filters need to be designed using the PCWLS method [12].

Define the transfer function of FIR FD filters $F(z, d_l)$ as

$$F(z, d_l) = \sum_{i=0}^{L-1} f_i(d_l)z^{-i} = f(d_l)\sum_{i=0}^{L-1} z^{-i}\phi_{ic}^{(S)}(z) \quad l = 0, \ldots, L$$

(5)

where $f(d_l) = [f_0(d_l), f_1(d_l), \ldots, f_{L-1}(d_l)]^T$. A complex error for each FIR FD filter design problem can be defined as

$$e_{FIR,l}(\omega) = H_z(e^{j\omega}, d_l) - F(e^{j\omega}, d_l) \quad l = 0, 1, \ldots, L$$

(6)

The weighted least-squares (WLS) error function can then be expressed as

$$E_{FIR,l} = \int_0^\pi W_1(\omega)|e_{FIR,l}(\omega)|^2 d\omega$$

(7)

$$= f(d_l)^T V_l f(d_l) - 2 f(d_l)^T v_l + \text{constant}$$

$$l = 0, 1, \ldots, L$$

where $W_1(\omega)$ is a given nonnegative weighting function and

$$V_l = \int_0^\pi W_1(\omega)\text{Re}\{\phi_{ic}^{(S)}(e^{j\omega})\phi_{ic}^{(S)*}(e^{j\omega})\} d\omega$$

(8)

$$v_l = \int_0^\pi W_1(\omega)\text{Re}\{e^{j\omega}\phi_{ic}^{(S)}(e^{j\omega})\} d\omega$$

(9)

In our designs, $W_1(\omega)$ assumes the same value for different fraction delay $d_l$. In order to control the peaks of such ripples, peak constraints are incorporated

$$\text{min } f_l^T V_l f_l - 2 f_l^T v_l$$

s.t. $|e_{FIR,l}(\omega)| \leq \delta$  \hspace{1cm} (10.a)

In (10.a), $\delta$ is a prescribed peak error limit. Essentially speaking, (10) is a quadratically constrained quadratic programming problem. Here, we adopt the project least-squares (PLS) approach [13] to solve the FIR FD filter design problem (10). However, the PLS approach can only deal with the linearly constrained quadratic programming problem. Therefore, the quadratic constraint (10.a) needs to be linearized. An approximation technique is applied here, which utilizes a $(2S)$-vertex regular polygon to approximate the circle $\text{Re}\{e_{FIR,l}(\omega)\} + \text{Im}\{e_{FIR,l}(\omega)\} \leq \delta$ as

$$\begin{align*}
\cos \left(\frac{\pi}{S}\right) \text{Re}\{e_{FIR,l}(\omega)\} + \sin \left(\frac{\pi}{S}\right) \text{Im}\{e_{FIR,l}(\omega)\} \\
\leq \delta \cos \left(\frac{\pi}{2S}\right)
\end{align*}$$

(11)

$$s = 0, \ldots, S-1 \quad l = 0, 1, \ldots, L \quad \omega \in [0, \pi] \quad i = 0, \ldots, J$$

In (11), $\text{Im}\{\} \text{ represents the imaginary part of a complex variable. With a sufficient large $S$, a close approximation can be obtained. By replacing (10.a) by the linear inequality constraint (11), (10) is transformed into a linearly constrained quadratic programming problem, which can be solved by the PLS approach.

B. Fixed Denominator Design

After obtaining the $L+1$ FIR FD filters, IIR FD filters with the fixed denominator are to be designed to best approximate these FIR FD filters in the WLS sense. The approximation error can be expressed as

$$E_{FIR,l} = \frac{1}{2\pi} \int_0^\pi W_1(\omega)|F(e^{j\omega}, d_l) - H(e^{j\omega}, d_l)|^2 d\omega$$

(12)

$$l = 0, 1, \ldots, L$$

In (12), $W_2(\omega)$ is a prescribed nonnegative weighting function, and $W_2(\omega) = W_2(-\omega)$. Like $W_1(\omega)$, $W_2(\omega)$ is set to be the same for different FD value $d_l$.

For designing an IIR VDF filter with a fixed denominator, an average FIR filter $F_{av}(z)$ is to be first determined in the time-domain from the FIR FD filters obtained in the previous step. The average FIR filter can be determined by

$$F_{av}(z) = \frac{1}{L+1} \sum_{l=0}^{L} F(z, d_l)$$

(13)

Then, a model reduction method [10] is utilized to obtain the fixed denominator $Q(z)$. The advantages of this model reduction method include: (a) The method is applicable to any IIR digital filter (including high-order) with a small numerical error since calculation is performed directly on filter coefficients without involving a transformation. (b) All the roots of the designed denominator will lie inside the unit circle if the procedure converges, which guarantees the stability of the designed IIR FD filter. (c) The model reduction method first determines the denominator. Except the numerator order $N$, the denominator design does not need any other information of the numerator, which greatly facilitates the design procedure. The iterative scheme of the model reduction method is given below:

1) Set $s = 0$, and choose $Q^{00}(z) = 1$.

2) Set $s = s+1$. Then, calculate $X^{s}(z)$ which is defined as

$$X^{s}(z) = z^{-T} G_s(z^{-T})z^{-T} F_{av}(z^{-T})^* Q^{-1}(z)$$

where $G_s(z^{-T})$ can be obtained by replacing the coefficients of $G(z^T)$ with their complex conjugates, where $G(z)$ is a maximum-phase polynomial (in $z^{-T}$) of order $T \geq N_0 = M-N-1$ whose squared magnitude is required to be equal to the specified $W_2(\omega)$, i.e., $W_2(\omega) = |G(e^{j\omega})|^2$.

3) Using $x^{(s)}_i$, construct the matrix $A^{(s)}$ and the vector $b^{(s)}$, which are defined as

$$A^{(s)} = \begin{bmatrix}
x^{(s)}_0 & 0 & \cdots & 0 \\
x^{(s)}_1 & x^{(s)}_0 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
x^{(s)}_{M-1} & \cdots & \cdots & x^{(s)}_0 \\
x^{(s)}_{M+N-1} & \cdots & \cdots & \cdots \\
x^{(s)}_{M+N-1} & \cdots & \cdots & \cdots 
\end{bmatrix}$$

(15)

$$b^{(s)} = \begin{bmatrix}
x^{(s)}_0 \cdots x^{(s)}_{M+N-1} \\
0 \cdots x^{(s)}_0 \cdots x^{(s)}_{M+N-1} \\
\vdots \cdots \vdots \cdots \cdots \\
0 \cdots x^{(s)}_0 \cdots x^{(s)}_{M+N-1} 
\end{bmatrix}$$

(16)
4) Calculate the denominator coefficients \( q_n^{(i)} \), which can minimize the approximation error \( e_{IR}^{(i)} \) defined as
\[
e_{IR}^{(i)} = \left[ A^{(i)} \cdot \bar{P}_M^{(i)} - b^{(i)} \right] \cdot \left[ A^{(i)} \cdot \bar{P}_M^{(i)} - b^{(i)} \right]^{\frac{1}{2}}
\]
where \( \bar{P}_M^{(i)} = [q_M^{(i)}, q_{M-1}^{(i)}, \ldots, q_2^{(i)}, q_1^{(i)}]^T \).

5) Repeating Steps 2 to 4 for a sufficient number of iterations, a minimum \( e_{IR}^{(i)} \) can be reached, and the entries of \( \bar{P}_M^{(i)} \) are chosen as the final denominator coefficients.

It has been proved in [11] that the roots of the resulting denominator always locate inside the unit circle if the iterative procedure converges.

C. Numeator Design

By taking \( G(z) = Q(z) \) into (12), \( E_{IR,j} \) can be rewritten as
\[
E_{IR,j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(\omega) \left[ F(e^{j\omega}, d_l) - H(e^{j\omega}, d_l) \right] \right|^2 d\omega
\]
\[l = 0, 1, \ldots, L\]
Note that given the fixed denominator \( Q(z) \), \( G(z)H(z,d_l) \) can be represented as a linear combination of the basis functions
\[
\eta_l(z) = \frac{z^{-\lambda}G(z)}{Q(z)} \lambda = 0, 1, \ldots, N
\]

Thus, the numerator design problem can be regarded as a linear approximation problem. According to the orthogonality principle, for a given \( Q(z) \), \( E_{IR,j} \) achieves its minimum value if and only if \( G(z)[F(z,d_l) - H(z,d_l)] \) is orthogonal to \( \eta_l(z) \), i.e.,
\[
\int_{-\pi}^{\pi} G(e^{j\omega}) \left[ F(e^{j\omega}, d_l) - \frac{p_l(e^{j\omega}, d_l)}{Q(e^{j\omega})} \right] \eta_l(e^{j\omega}) d\omega = 0
\]
By solving the linear equation in (20), we can obtain the optimal numerator for a given \( Q(z) \).

D. VFD Polynomial Coefficients Approximation

Based on (3), the polynomial fitting error functions are defined as
\[
E_n = \sum_{j=0}^{N} \left( \sum_{k=0}^{N} a_n(k)d_l^k - p_n(d_l) \right)^2 \quad n = 0, \ldots, N
\]
The optimal polynomial coefficients \( a_n(k) \) for given \( p_n(d_l) \) can be obtained by setting to zero the partial derivative of \( E_n \) with respect to each polynomial coefficient, i.e., \( \partial E_n/\partial a_n(k) = 0 \).

IV. SIMULATIONS

Three sets of filter-examples are presented to illustrate the effectiveness of the proposed method. In order to evaluate the performance of designed IIR VFD filters, the maximum absolute error (MAE) and the L2 error (L2E) are defined as
\[
e_{MA,j} = \max \left[ e_{abs,j}(\omega, d), \omega \in [0, w_c], |d| \leq 0.5 \right] \quad j = 1, 2
\]
\[
e_{L2,j} = \int_{0}^{\omega_c} \int_{-d}^{d} e_{abs,j}(\omega, p) dp d\omega \quad j = 1, 2
\]
where
\[
e_{abs,1}(\omega, d) = \left| H(e^{j\omega}, d) - H(e^{j\omega}, d) \right|
\]
\[
e_{abs,2}(\omega, d) = \left| \tau(\omega, d) - (D + d) \right|
\]
In (25), \( \tau(\omega, d) \) denotes the group delay of a designed VFD filter. In all the filter-examples, the polynomial order is set to \( K = 6 \). The total number of FD filters designed in the first step is always chosen as 11, i.e., \( L = 10 \). The peak error limit \( \delta \) used in (11) is set to \( 10^{-3} \) in all the filter-examples. For the PCWLS FIR FD filter design, \( S \) in (11) is always chosen as 32.

The specifications of the IIR VFD filters are shown in Table I. For each of three sets of \( L_{FD} \) and \( D \) values, five pairs of filter orders (i.e., \( N \) and \( M \)) and three different \( w_c \) are employed. The weighting function \( W_1(\omega) \) in (7) is chosen as \( W_1(\omega) = 1 \) within \([0, w_c]\) and \( W_1(\omega) = 0 \) within \([w_c, \pi]\). The weighting function \( W_2(\omega) \) in (12), which is used to determine \( G(z) \), is chosen as \( W_2(\omega) = 1 \) for \( |\omega| \leq w_c \) and \( W_2(\omega) = 10^{-20} \) for \( |\omega| < |\omega| < \pi \). After obtaining the polynomial coefficients, the L2 protein (for \( r = 2 \) and 16) FD filters are obtained using (3) to evaluate the interpolation ability of the proposed method.

The magnitude and group delay responses of some IIR FD filters with \( r = 2 \) are shown in Figs. 1-3. It can be observed from the design results that the performance for FD filters with different \( r \) (1, 2, 16) are similar and consistent which confirms the ability of such IIR VFD filters to interpolate. Design results confirm that all the designed IIR VFD filters are stable. In the first set of filter-examples (with \( L_{FD} = 80 \) and \( D = 21 \)), as \( w_c \) decreases from 0.95\( \pi \) to 0.9\( \pi \), the designed IIR VFD filters can achieve reduced MAE and L2E values of magnitude and group delay responses. Also, the best combination of \((N, M)\) for \( w_c = 0.95\pi, 0.925\pi, 0.9\pi \), respectively, are \((40, 10), (41, 4), (41, 4)\). In the second set of filter-examples (with \( L_{FD} = 60 \) and \( D = 16 \)), as \( w_c \) decreases from 0.875\( \pi \) to 0.825\( \pi \), the designed IIR VFD filters with \( w_c = 0.85\pi \) achieve the best performance with \((N, M) = (30, 10)\). The best combination of \((N, M)\) for \( w_c = 0.875\pi \) is \((31, 6)\) whereas the best combinations of \((N, M)\) for \( w_c = 0.825\pi \) are \((30, 8)\) for reduced L2E in magnitude and group delay responses and \((31, 4)\) for reduced MAE in magnitude and group delay responses. In the third set of filter-examples (with \( L_{FD} = 40 \) and \( D = 11 \)), as \( w_c \) decreases from 0.8 to 0.75\( \pi \), the MAE and L2E values of magnitude and group delay response increase, and the best combination of \((N, M)\) among these three \( w_c \) values is \((21, 6)\). The increasing trends of the MAE and L2E values of magnitude and group delay responses of the first and second sets of filter-examples are opposite of each other while that of the second set of filter-examples exhibit minimum MAE and L2E values when the \( w_c \) value is at the middle. From the results, it can be observed that: (a) For \( L_{FD} = 80 \) and \( D = 21 \), the best overall performance can be achieved for \((N, M) = (40, 10)\) and \( w_c = 0.9\pi \). (b) For \( L_{FD} = 60 \) and \( D = 16 \), the best overall performance can be achieved for \((N, M) = (30, 10)\) and \( w_c = 0.85\pi \). (c) For \( L_{FD} = 40 \) and \( D = 11 \), the best overall performance can be achieved for \((N, M) = (21, 6)\) and \( w_c = 0.8\pi \).

### Table I. IIR VFD Filter Specifications

<table>
<thead>
<tr>
<th>( w_c, \pi )</th>
<th>( L_{FD} )</th>
<th>( \delta )</th>
<th>((N, M))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0.95, 0.925, 0.9])</td>
<td>80</td>
<td>([40, 10, 41, 4, 41, 4])</td>
<td>([41, 2])</td>
</tr>
<tr>
<td>([0.875, 0.85, 0.825])</td>
<td>60</td>
<td>([30, 10, 31, 4])</td>
<td>([31, 2])</td>
</tr>
<tr>
<td>([0.8, 0.775, 0.75])</td>
<td>40</td>
<td>([20, 10, 21, 6])</td>
<td>([21, 4])</td>
</tr>
</tbody>
</table>

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V. CONCLUSIONS

In this paper, the methodology and performance of IIR VFD filters designed using a two-stage scheme and realizable via the Farrow’s structure have been described, analyzed, and presented. IIR VFD filters with a low-order fixed denominator offer the following advantages: (i) Satisfactory MAE and L2E performance in magnitude and group delay responses due to a precise control by the denominator over its passband but the required denominator filter order is low (and within $M = 4$ to 10 in the filter-examples). (ii) IIR VFD filters can be designed to exhibit low group delay value. (iii) A fixed denominator IIR VFD filter can be easily designed to be stable by model reduction of an FIR obtained through time-domain averaging of the $L+1$ FIR FD filter coefficients; and does not introduce undesirable transients as in the case of a variable denominator IIR VFD filter. (iv) IIR VFD filters offer a flexible combination of the numbers of coefficients in the variable numerator and the fixed denominator tailored to any specified set of passband cutoff frequency $w_c$ and MAE and L2E values in magnitude and group delay responses. Details of a comparison of IIR VFD filter design with those of corresponding FIR and allpass VFD filter designs are to be reported in a forthcoming paper in the IEEE Transactions on Circuits and Systems I.

VI. REFERENCES


Figure 1. IIR VFD filters ($LFIR = 80, N = 40, M = 10, w_c = 0.9\pi, r = 2$).

Figure 2. IIR VFD filters ($LFIR = 60, N = 30, M = 10, w_c = 0.85\pi, r = 2$).

Figure 3. IIR VFD filters ($LFIR = 40, N = 21, M = 6, w_c = 0.8\pi, r = 2$).