

Quantum Computation in $\otimes^n(\text{APS})P_3$

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1 Introduction

Quantum effects are certain to become more important in computer chips as processing power and storage density continue to increase more or less in accordance with Moore's law. In classical computers, quantum effects are a source of errors and place a limit on possible miniaturization. The promise of quantum computation is an ability to exploit quantum effects to create computational engines of vastly increased power for some types of calculation. Security agencies are interested in quantum computation because of its potential ability to break cryptographic codes. Physicists enjoy work in the area because of its ties to the foundations of quantum theory, especially the measurement process.

In classical computation, information is represented in collections of bits, each of which can take values of either 0 or 1. In quantum computation[1], we use instead *qubits*, each of which can be in a linear superposition

$$|1\rangle e^{-i\phi/2} \cos \theta/2 + |0\rangle e^{i\phi/2} \sin \theta/2$$

of states $|0\rangle$ and $|1\rangle$. Any pure state of a qubit corresponds to a unique spin-1/2 direction (θ, ϕ) in space. The qubits and the quantum gates that act on them are efficiently and conveniently represented in the *algebra of physical space* (APS).[2, 3] Efficiency is important, since any practical quantum computer will need thousands of qubits, and treatments must be able to compute decoherence effects and mixed states.

In quantum computation, linear superpositions of n qubits evolve in a way that is reversible except at evaluation steps and during decoherence

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processes. The reversible time evolution is given by unitary transformations and represents a massively parallel computation of all possible states of the n -qubit system. To simulate the evolution efficiently on classical computers, we plan to apply parallel computations in SHARCNET. At the present, we are still developing Maple and C code for computations with several qubits.

Here we illustrate the algebra for systems of two qubits and study the question of how to define entanglement in mixed-state systems.

2 Rotations in APS

APS is Clifford's geometric algebra[4, 5, 6] of vectors in a three-dimensional Euclidean space. Elements of the algebra can be represented by 2×2 matrices in which the orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are represented by Pauli spin matrices. In fact, there are an infinite number of matrix representations, but they all share the same algebra, namely APS. The one axiom that determines the structure of the algebra gives the meaning of the product of a vector with itself:

$$\mathbf{v}\mathbf{v} \equiv \mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v} \quad (1)$$

In particular, $\mathbf{e}_j^2 = 1$. It also follows (put $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2$) that perpendicular vectors anticommute, for example

$$\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0.$$

The product $\mathbf{e}_1\mathbf{e}_2$ is a *bivector* for the plane spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$; it generates rotations in the plane. To rotate \mathbf{e}_3 to an arbitrary direction \mathbf{s} in space, we apply

$$\mathbf{s} = R\mathbf{e}_3R^\dagger, \quad (2)$$

where the *rotor* R is $R = \exp\left(\theta\hat{\mathbf{B}}/2\right)$ and θ is the angle of rotation from \mathbf{e}_3 to \mathbf{s} in the plane of the unit bivector $\hat{\mathbf{B}}$, which like $\mathbf{e}_1\mathbf{e}_2$, squares to -1 . The dagger conjugation is called *reversion*: it reverses the order of vector factors, for example

$$(\mathbf{e}_1\mathbf{e}_2)^\dagger = \mathbf{e}_2\mathbf{e}_1.$$

Reversion is equivalent to hermitian conjugation in the standard matrix representation of the algebra. Rotors are *even* elements of the algebra: linear combinations of products of bivectors. Because bivectors are antithermitian, rotors are also unitary: $R^\dagger = R^{-1}$. Note that the volume element $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

of APS commutes with all elements and squares to -1 . It can therefore be associated with the unit imaginary:

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = i ,$$

and bivectors can be expressed as imaginary vectors (pseudovectors) whose spatial direction gives the normal to the plane, for example

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_3 = i\mathbf{e}_3 .$$

Rotors are elements of the spin representation $Spin(3) \simeq SU(2)$ of rotations. The product of rotations is represented by the rotor product, say R_2R_1 . Rotors are basic elements in a simple geometric representation of spatial rotations as vectors on the unit sphere, in which products of rotations are given by the nonabelian addition of such vectors[7]. The *irreducible spinors* that carry the representation are formed by applying the rotor to a *projector* (a real idempotent) such as

$$P_3 = \frac{1}{2}(1 + \mathbf{e}_3) = P_3^2 = P_3^\dagger.$$

A typical spinor of this sort is RP_3 . The spinor space lies in the minimum left ideal $(\text{APS})P_3$ of the algebra. The irreducible spinor representation is faithful since the rotor itself is readily obtained as twice the even part of RP_3 . The projector complementary to P_3 is the *Clifford conjugate*

$$\begin{aligned} \bar{P}_3 &= \frac{1}{2}(1 - \mathbf{e}_3) = 1 - P_3 \\ P_3\bar{P}_3 &= \bar{P}_3P_3 = 0. \end{aligned}$$

Spinors $R\bar{P}_3$ of the complementary minimal left ideal $(\text{APS})\bar{P}_3$ carry an inequivalent irreducible representation of $SU(2)$.

The component of the direction \mathbf{s} [Eq.(2)] in the direction of the unit vector \mathbf{m} is readily expressed in terms of the spinor $RP_3 \equiv \psi$

$$\begin{aligned} \mathbf{s} \cdot \mathbf{m} &= \langle \mathbf{s}\mathbf{m} \rangle_S = \langle R\mathbf{e}_3R^\dagger\mathbf{m} \rangle_S \\ &= \langle R(P_3 - \bar{P}_3)R^\dagger\mathbf{m} \rangle_S \\ &= \langle RP_3R^\dagger\mathbf{m} \rangle_S - \langle R\bar{P}_3R^\dagger\mathbf{m} \rangle_S \\ &= 2\langle P_3R^\dagger\mathbf{m}RP_3 \rangle_S \\ &= 2\langle \psi^\dagger\mathbf{m}\psi \rangle_S , \end{aligned}$$

where $\langle x \rangle_S = (x + \bar{x})/2$ is the scalar part of x , and we noted $\langle ab \rangle_S = \langle ba \rangle_S = \langle \overline{ab} \rangle_S$. This is seen to have the usual quantum form if we note that $2\langle x \rangle_S$ is equivalent to the trace of the matrix representation of x , and the matrix representation of \mathbf{m} is commonly written $\sigma \cdot \mathbf{m}$.

2.1 Spinor States and Spin-State Densities

The spinor $\psi = RP_3$ represents the pure state with the spin direction \mathbf{s} as given by Eq.(2). In terms of Euler angles ϕ, θ, χ , about space-fixed axes, the rotor R can be expressed by

$$\begin{aligned} R &= \exp(-i\mathbf{e}_3\phi/2) \exp(-i\mathbf{e}_2\theta/2) \exp(-i\mathbf{e}_3\chi/2) \\ &= \exp(-i\mathbf{n}\theta/2) \exp[-i\mathbf{e}_3(\phi + \chi)/2] \\ &= \left(\cos \frac{\theta}{2} - i\mathbf{n} \sin \frac{\theta}{2} \right) \exp[-i\mathbf{e}_3(\phi + \chi)/2], \end{aligned} \quad (3)$$

where $\mathbf{n} = \exp(-i\mathbf{e}_3\phi/2) \mathbf{e}_2 \exp(i\mathbf{e}_3\phi/2)$ is a unit vector in the $\mathbf{e}_1\mathbf{e}_2$ plane and $-i\mathbf{n} = \exp(-i\mathbf{n}\pi/2)$ is the rotor for a 180° rotation about \mathbf{n} . Therefore, any rotor R is a real linear combination $\cos \frac{\theta}{2} R_\uparrow + \sin \frac{\theta}{2} R_\downarrow$ of two independent rotors $R_\uparrow = \exp[-i\mathbf{e}_3(\phi + \chi)/2]$ and $R_\downarrow = -i\mathbf{n}R_\uparrow$ that are mutually orthogonal: $\langle R_\uparrow R_\downarrow^\dagger \rangle_S = \langle -i\mathbf{n} \rangle_S = 0$.

By projecting the rotors with P_{+3} we obtain the corresponding relation for ideal spinors:

$$\begin{aligned} \psi &\equiv RP_{+3} = \cos \frac{\theta}{2} \psi_\uparrow + \sin \frac{\theta}{2} \psi_\downarrow \\ \psi_\uparrow &\equiv R_\uparrow P_{+3} = e^{-i(\phi+\chi)/2} P_{+3}, \quad \psi_\downarrow \equiv R_\downarrow P_{+3} = -i\mathbf{n} \psi_\uparrow. \end{aligned} \quad (4)$$

In the last line, we used the ‘‘pacwoman’’ property of projectors $\mathbf{e}_3 P_3 = P_3$ to equate

$$e^{-i(\phi+\chi)\mathbf{e}_3/2} P_{+3} = e^{-i(\phi+\chi)/2} P_{+3}.$$

Note that ψ is an eigenspinor of \mathbf{s} with eigenvalue $+1$:

$$\mathbf{s}\psi = R\mathbf{e}_3 R^\dagger RP_3 = \psi. \quad (5)$$

Similarly, the spinors ψ_\uparrow and ψ_\downarrow are eigenfunctions of \mathbf{e}_3 with eigenvalues $+1$ (for ‘‘spin up’’) and -1 (for ‘‘spin down’’):

$$\mathbf{e}_3 \psi_\uparrow = \psi_\uparrow, \quad \mathbf{e}_3 \psi_\downarrow = -\psi_\downarrow.$$

Evidently \mathbf{e}_3 is the spin operator that is customarily taken to represent the z component $\mathbf{s} \cdot \mathbf{e}_3$ of the spin in units of $\hbar/2$ for the P_3 ideal representation of ψ . In APS, it is clear that any unit vector \mathbf{s} has an eigenspinor $\psi \in (\text{APS})P_3$ satisfying (5) that represents a spin in the direction of \mathbf{s} . Traditional orthonormality conditions hold:

$$\begin{aligned} 2 \langle \psi_{\uparrow} \psi_{\downarrow}^{\dagger} \rangle_S &= 2 \langle \psi_{\uparrow} \psi_{\uparrow}^{\dagger} i \mathbf{n} \rangle_S = 2 \langle P_{+3} i \mathbf{n} \rangle_S = 0 \\ 2 \langle \psi_{\downarrow} \psi_{\downarrow}^{\dagger} \rangle_S &= 2 \langle \psi_{\uparrow} \psi_{\uparrow}^{\dagger} \rangle_S = 2 \langle P_{+3} \rangle_S = 1. \end{aligned}$$

It follows that the amplitudes for measuring the spin direction of the pure state ψ to be up or down along \mathbf{e}_3 are

$$\begin{aligned} 2 \langle \psi \psi_{\uparrow}^{\dagger} \rangle_S &= \cos \frac{\theta}{2} \\ 2 \langle \psi \psi_{\downarrow}^{\dagger} \rangle_S &= \sin \frac{\theta}{2} \end{aligned}$$

giving probabilities

$$\left| 2 \langle \psi \psi_{\uparrow}^{\dagger} \rangle_S \right|^2 = \cos^2 \frac{\theta}{2} = \frac{1}{2} (1 + \mathbf{s} \cdot \mathbf{e}_3) \quad (6)$$

$$\left| 2 \langle \psi \psi_{\downarrow}^{\dagger} \rangle_S \right|^2 = \sin^2 \frac{\theta}{2} = \frac{1}{2} (1 - \mathbf{s} \cdot \mathbf{e}_3). \quad (7)$$

These probabilities can equivalently be computed from the *spin-state density* for the pure state ψ ,

$$\varrho = \psi \psi^{\dagger} = R P_3 R^{\dagger} = \frac{1}{2} (1 + \mathbf{s}) \quad (8)$$

as $2 \langle \varrho P_3 \rangle_S$ and $2 \langle \varrho \bar{P}_3 \rangle_S$, respectively. The spin-state density of a pure state is seen to be a projector for the direction \mathbf{s} of the spin.

A linear combination of orthogonal pure spin-state densities ϱ^j is a *mixed* spin-state density

$$\varrho = \sum_j w_j \varrho^j, \quad \sum_j w_j = 1.$$

This is a classical mixture of pure states, in which the fraction of systems in pure state j is w_j . One measure of the purity of the state is

$$2 \langle \varrho^2 \rangle_S = 2 \left\langle \left(\sum_j w_j \varrho^j \right)^2 \right\rangle_S = 2 \left\langle \sum_j w_j^2 \varrho^j \right\rangle_S = \sum_j w_j^2,$$

which is unity for a pure state and $1/2$ for a fully mixed (unpolarized) system. We used the fact here that the spin-state densities of pure states are projectors, and there are only two independent spinor states ($j = 1, 2$) for the one-particle systems considered.

The expressions for the probabilities of measurement are easily generalized. Consider two pure states with spinors $\psi_1 = R_1 P_3$, $\psi_2 = R_2 P_3$ and corresponding spin-state densities $\varrho_1 = R_1 P_3 R_1^\dagger = \frac{1}{2}(1 + \mathbf{s}_1)$ and $\varrho_2 = R_2 P_3 R_2^\dagger = \frac{1}{2}(1 + \mathbf{s}_2)$. If a system with spin direction \mathbf{s}_1 (“in state ψ_1 ”) is split by a measurement into spin directions $\pm \mathbf{s}_2$, the fraction in the \mathbf{s}_2 direction is

$$\begin{aligned}
2 \langle \varrho_1 \varrho_2 \rangle_S &= 2 \left\langle R_1 P_3 R_1^\dagger R_2 P_3 R_2^\dagger \right\rangle_S \\
&= 2 \left\langle \left(P_3 R_1^\dagger R_2 P_3 \right) \left(P_3 R_2^\dagger R_1 P_3 \right) \right\rangle_S \\
&= 2 \left\langle 2 \left\langle P_3 R_1^\dagger R_2 P_3 \right\rangle_S P_3 2 \left\langle P_3 R_2^\dagger R_1 P_3 \right\rangle_S \right\rangle_S \\
&= \left| 2 \left\langle \psi_1 \psi_2^\dagger \right\rangle_S \right|^2 = \frac{1}{2} (1 + \mathbf{s}_1 \cdot \mathbf{s}_2). \tag{9}
\end{aligned}$$

This expression is easily generalized further by linearity to the measurement of mixed states.

From the Euler-angle expression (3) for R , it is easy to see that ψ has the standard matrix representation

$$\psi \equiv R P_{+3} = e^{-i\chi/2} \begin{pmatrix} e^{-i\phi/2} \cos \theta/2 & 0 \\ e^{i\phi/2} \sin \theta/2 & 0 \end{pmatrix}.$$

If the column of zeros is dropped, the standard matrix representation of ψ is the two-component spinor familiar from the usual nonrelativistic Pauli theory.

3 Bell’s Theorem

Quantum systems are often tested against Bell’s theorem. Violation of Bell’s inequalities is often used as evidence of entanglement (nonlocal quantum correlation). A simple proof of the inequalities can be derived[8] from polarization splitters acting on a single beam. The nice aspect of this derivation is that the question of nonlocality does not arise. It shows that Bell’s inequality concerns directly the difference between classical probabilities for particles and quantum probability amplitudes in waves. Nonlocality arises

only in EPR-type applications with entangled states, but the inequality itself is violated locally for a single beam of waves.

Consider a beam of silver atoms passing through a sequence of Stern-Gerlach magnets aligned to split the beam into spins polarized in opposite directions specified by unit vectors $\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{c}$. If one of the polarized beams leaving a given Stern-Gerlach magnet is blocked, that magnet serves as a polarization filter. Let $f(\mathbf{c}, \mathbf{b}, \mathbf{a})$ be the fraction of atoms passing through sequential polarization filters of types $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Then, with “classical” atoms (the same math works for polarized photons), the fraction that pass through filters \mathbf{a} and then \mathbf{c} is just the sum of those that pass through $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}, -\mathbf{b}, \mathbf{c}$:

$$f(\mathbf{c}, \mathbf{a}) = f(\mathbf{c}, \mathbf{b}, \mathbf{a}) + f(\mathbf{c}, -\mathbf{b}, \mathbf{a}).$$

Similarly,

$$\begin{aligned} f(-\mathbf{c}, \mathbf{b}) &= f(-\mathbf{c}, \mathbf{b}, \mathbf{a}) + f(-\mathbf{c}, \mathbf{b}, -\mathbf{a}) \\ f(\mathbf{c}, \mathbf{b}, \mathbf{a}) + f(-\mathbf{c}, \mathbf{b}, \mathbf{a}) &= f(\mathbf{b}, \mathbf{a}). \end{aligned}$$

Adding these three relations we obtain

$$\begin{aligned} f(\mathbf{c}, \mathbf{a}) + f(-\mathbf{c}, \mathbf{b}) &= f(\mathbf{b}, \mathbf{a}) + f(\mathbf{c}, -\mathbf{b}, \mathbf{a}) + f(-\mathbf{c}, \mathbf{b}, -\mathbf{a}) \\ &\geq f(\mathbf{b}, \mathbf{a}) \end{aligned} \tag{10}$$

which is a form of Bell’s inequality. We found above (9) that $f(\mathbf{c}, \mathbf{a}) = \frac{1}{2}(1 + \mathbf{c} \cdot \mathbf{a})$ so that Bell’s inequality (10) can be written

$$1 + \mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} \geq \mathbf{b} \cdot \mathbf{a}.$$

If we let \mathbf{b} bisect \mathbf{a} and \mathbf{c} , so that we can put $\mathbf{c} \cdot \mathbf{a} = \cos 2\theta$ and $\mathbf{c} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \cos \theta$, we obtain

$$1 + \cos 2\theta \geq 2 \cos \theta.$$

However, this inequality is broken for $\theta = \pi/4$, for example. The error is in working with fractions or probabilities instead of amplitudes. Our method indeed gives

$$\begin{aligned} f(\mathbf{c}, \mathbf{a}) &= 2 \langle P_{\mathbf{c}} P_{\mathbf{a}} P_{\mathbf{c}} \rangle_S = \frac{1}{2} (1 + \mathbf{a} \cdot \mathbf{c}) \\ &= 2 \langle P_{\mathbf{c}} (P_{\mathbf{b}} + \bar{P}_{\mathbf{b}}) P_{\mathbf{a}} (P_{\mathbf{b}} + \bar{P}_{\mathbf{b}}) P_{\mathbf{c}} \rangle_S, \end{aligned}$$

which differs from the classical result

$$\begin{aligned}
f(\mathbf{c}, \mathbf{b}, \mathbf{a}) + f(\mathbf{c}, -\mathbf{b}, \mathbf{a}) &= 2 \langle \mathbf{P}_c \mathbf{P}_b \mathbf{P}_a \mathbf{P}_b \mathbf{P}_c \rangle_S + 2 \langle \mathbf{P}_c \bar{\mathbf{P}}_b \mathbf{P}_a \bar{\mathbf{P}}_b \mathbf{P}_c \rangle_S \\
&= \langle \mathbf{P}_c \mathbf{P}_b \mathbf{P}_c \rangle_S (1 + \mathbf{b} \cdot \mathbf{a}) \\
&\quad + \langle \mathbf{P}_c \bar{\mathbf{P}}_b \mathbf{P}_c \rangle_S (1 - \mathbf{b} \cdot \mathbf{a}) \\
&= \frac{1}{2} (1 + \mathbf{b} \cdot \mathbf{c} \mathbf{b} \cdot \mathbf{a})
\end{aligned}$$

unless $\mathbf{b} = \mathbf{a}$ or $\mathbf{b} = \mathbf{c}$.

In the usual application to an entangled state in an EPR-type measurement, some of the measurements are performed on one subsystem and some on the other, and for the singlet state, a measurement of polarization \mathbf{n} in one subsystem is equivalent to a measurement of $-\mathbf{n}$ in the other.

4 Qubits

The simplest example of a two-level quantum system is the spin-1/2 system considered above. The same formalism can be applied to qubits. We first consider operations on single spin systems in the language of qubits and quantum computing, and next present some extensions to systems of two or more qubits.

4.1 Qubit Operators

There is arbitrariness in the phase as well as in factors of \mathbf{e}_3 in ψ_\uparrow and ψ_\downarrow . Let us therefore define a *standard ideal spinor basis*, which we denote in the ket notation

$$\begin{aligned}
|\uparrow\rangle &= \mathbf{P}_3 \\
|\downarrow\rangle &= \mathbf{e}_1 |\uparrow\rangle = \mathbf{e}_1 \mathbf{P}_3 |\uparrow\rangle = \mathbf{e}_1 \mathbf{e}_3 \mathbf{P}_3 |\uparrow\rangle = -i \mathbf{e}_2 \mathbf{P}_3 |\uparrow\rangle .
\end{aligned}$$

The *unipotent* ($\mathbf{e}_1^2 = 1$) element \mathbf{e}_1 interchanges these: $|\downarrow\rangle = \mathbf{e}_1 |\uparrow\rangle$ and $|\uparrow\rangle = \mathbf{e}_1 |\downarrow\rangle$. We may think of the “up” state $|\uparrow\rangle$ as representing the bit value 1, and the “down” state $|\downarrow\rangle$ as the bit value 0. Evidently, \mathbf{e}_1 acts as the NOT operator. It is also a *reflection operator*, since for any vector \mathbf{v}

$$\mathbf{v} \rightarrow \mathbf{e}_1 \mathbf{v} \mathbf{e}_1 = \mathbf{v}^\parallel - \mathbf{v}^\perp = 2\mathbf{v} \cdot \mathbf{e}_1 \mathbf{e}_1 - \mathbf{v}$$

keeps the component \mathbf{v}^\parallel of \mathbf{v} along \mathbf{e}_1 constant and reverses components perpendicular to \mathbf{e}_1 . The dual to \mathbf{e}_1 is $-i\mathbf{e}_1 = \exp(-i\pi\mathbf{e}_1/2)$, a rotor for the rotation of π about \mathbf{e}_1 . If the rotor is applied twice, the result is a

2π rotation, which changes the sign of any spinor for a spin-1/2 system: $(-i\mathbf{e}_1)^2 = -1$. Note that any unit vector in the \mathbf{e}_{12} plane can serve as the NOT operator. The resultant states differ only in their relative phase.

A state in which the spin is aligned with \mathbf{e}_1 is an eigenstate of the NOT operator. Such a state is obtained by applying the rotor $\exp(-i\pi\mathbf{e}_2/4)$ to P_3 , which rotates the spin from \mathbf{e}_3 by $\pi/2$ in the $\mathbf{e}_3\mathbf{e}_1$ plane. It can be expressed in several ways:

$$e^{-i\pi\mathbf{e}_2/4}P_3 = \left(\frac{1-i\mathbf{e}_2}{\sqrt{2}}\right)P_3 = \left(\frac{1+\mathbf{e}_1}{\sqrt{2}}\right)P_3 = \left(\frac{\mathbf{e}_3+\mathbf{e}_1}{\sqrt{2}}\right)P_3 .$$

In the fourth form, we see the eigenstate of the NOT operator is also obtained by a reflection of the up basis state in the unit vector $\hat{\mathbf{h}} = (\mathbf{e}_1 + \mathbf{e}_3)/\sqrt{2} = e^{i\pi\mathbf{e}_2/4}\mathbf{e}_1 = e^{-i\pi\mathbf{e}_2/4}\mathbf{e}_3$. This reflection is called the *Hadamard transformation*.

Since any state can be expressed as a linear combination of the basis states P_3 and \mathbf{e}_1P_3 , we can find the effect of an operation on any state if we know its effect of the basis states. In the Hadamard transformation,

$$\begin{aligned} |\uparrow\rangle &\rightarrow \hat{\mathbf{h}}|\uparrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \\ |\downarrow\rangle &\rightarrow \hat{\mathbf{h}}|\downarrow\rangle = \hat{\mathbf{h}}\mathbf{e}_1|\uparrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} . \end{aligned}$$

4.2 Exponential Forms

Rotations can be written in exponential form with a scalar parameter proportional to the angle of rotation. They are thus connected to unity (scalar parameter = 0) and can easily be interpolated. Reflections, on the other hand, are usually viewed as discrete transformations, performed entirely or not at all. However, unit vectors can also be extended to exponential form. Such exponential forms are useful for working out physical interactions in the Hamiltonian that induce the reflections through the time evolution operator.

Consider first the dual $-i\mathbf{n}$ of the real unit vector \mathbf{n} . It is a rotor for a rotation by angle π about the \mathbf{n} axis and can be viewed as the continuous transformation $\exp(-i\mathbf{n}\alpha/2)$ when $\alpha = \pi$:

$$-i\mathbf{n} = \exp(-i\mathbf{n}\alpha/2), \quad \alpha = \pi .$$

The unit vector \mathbf{n} itself can be expressed in terms of the projector $P_{\mathbf{n}} =$

$\frac{1}{2}(1 + \mathbf{n})$:

$$\begin{aligned} \mathbf{n} &= i \exp(-i\mathbf{n}\pi/2) = \exp[i(1 - \mathbf{n})\pi/2] \\ &= \exp[i\bar{P}_\mathbf{n}\pi] = \exp[i\bar{P}_\mathbf{n}\alpha], \quad \alpha = \pi. \end{aligned}$$

In other words, it is part of a continuous transformation generated by the projector $\bar{P}_\mathbf{n}$. At intermediate angles α we note that

$$e^{i\alpha\bar{P}_\mathbf{n}} = e^{i\alpha\bar{P}_\mathbf{n}} (P_\mathbf{n} + \bar{P}_\mathbf{n}) = P_\mathbf{n} + e^{i\alpha}\bar{P}_\mathbf{n}.$$

4.3 Raising and Lowering Operators

We saw above that in the ideal-spinor representation, the unit vector \mathbf{n} is an operator that is customarily taken to represent the spin component in the direction \mathbf{n} in units of $\hbar/2$. One may verify that the commutation relations of unit vectors in space are just what one needs for components of the angular momentum, and of course that is why Pauli introduced his spin matrices to represent electron spin. The usual operator for the squared spin is thus

$$S^2 = \left(\frac{\hbar}{2}\right)^2 (\mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2),$$

which is immediately evaluated in APS to $\frac{3}{4}\hbar^2$. The squared spin in APS is more naturally taken to be

$$\left(\frac{\hbar}{2}\right)^2 \mathbf{s}^2 = \left(\frac{\hbar}{2}\right)^2.$$

The *raising* operator s_+ is defined by $s_+|\downarrow\rangle = |\uparrow\rangle$, $s_+|\uparrow\rangle = 0$. Any nonzero element that annihilates $P_3 = |\uparrow\rangle$ must have the form $x\bar{P}_3$, and $x\bar{P}_3|\downarrow\rangle = |\uparrow\rangle$ is satisfied by $x = \mathbf{e}_1$. Thus, we put

$$s_+ = \mathbf{e}_1\bar{P}_3 = \frac{1}{2}(\mathbf{e}_1 + i\mathbf{e}_2) = P_3\mathbf{e}_1. \quad (11)$$

Similarly, the lowering operator is $s_- = \mathbf{e}_1P_3 = s_+^\dagger$. The operators s_\pm are examples of nilpotent operators: $s_\pm^2 = 0$. Their products are $s_+s_- = P_3$, $s_-s_+ = \bar{P}_3$, which gives $s_+s_- + s_-s_+ = 1$ and $s_+s_- - s_-s_+ = \mathbf{e}_3$.

4.4 Two-Qubit Systems

We represent multi-qubit systems very generally as tensor products of single qubit systems, that is, in tensor products of ideals of APS. This representation is sufficiently general to handle distinguishable quantum systems. If the

systems are identical, additional symmetries apply. Tensor (or “Kronecker”) products obey the fundamental relation

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

We label the two qubits A and B . With two qubits or spin-1/2 systems, the projector is the tensor (or “Kronecker”) product $P_3 \otimes P_3$ and the general pure state can be written

$$\psi_{AB} = \Psi_{AB}(P_3 \otimes P_3).$$

If Ψ_{AB} is a *product state*, it can be expressed as a single product term $\Psi_{AB} = R_A \otimes R_B$ so that

$$\psi_{AB} = \Psi_{AB}(P_3 \otimes P_3) = (R_A \otimes R_B)(P_3 \otimes P_3) = R_A P_3 \otimes R_B P_3$$

and the spin-state density is then also a single product

$$\begin{aligned} \varrho_{AB} &= \psi_{AB} \psi_{AB}^\dagger = (R_A \otimes R_B)(P_3 \otimes P_3)(R_A \otimes R_B)^\dagger \\ &= (R_A \otimes R_B)(P_3 \otimes P_3)(R_A^\dagger \otimes R_B^\dagger) \\ &= (R_A P_3 R_A^\dagger) \otimes (R_B P_3 R_B^\dagger) \\ &= \varrho_A \otimes \varrho_B. \end{aligned}$$

Such systems, whose density operators can be expressed as a single tensor product of subsystem density operators, are *decomposable*. In a decomposable system, measurements in one subsystem are independent of those in another. There is no entanglement and indeed no correlation between the subsystems. Here, each subsystem spin-state density ϱ_A, ϱ_B , represents the pure state of a single qubit and therefore has the form of a single-spin projector (8). If we put $\mathbf{s}_A = R_A \mathbf{e}_3 R_A^\dagger$ and $\mathbf{s}_B = R_B \mathbf{e}_3 R_B^\dagger$, then the decomposable state is itself a primitive two-spin projector

$$\varrho_A \otimes \varrho_B = P_{\mathbf{s}_A} \otimes P_{\mathbf{s}_B} = \frac{1}{4}(1 + 1 \otimes \mathbf{s}_B + \mathbf{s}_A \otimes 1 + \mathbf{s}_A \otimes \mathbf{s}_B).$$

The projector is *primitive* because it projects onto a single 2-particle state and cannot be written as the sum of other projectors. It together with its 3 conjugate forms, $\bar{P}_{\mathbf{s}_A} \otimes P_{\mathbf{s}_B}, P_{\mathbf{s}_A} \otimes \bar{P}_{\mathbf{s}_B}$, and $\bar{P}_{\mathbf{s}_A} \otimes \bar{P}_{\mathbf{s}_B}$, split the two-spin space into four orthogonal parts. The sum of any two of the primitive projectors

gives a *simple projector*, which splits the spin space in half. There are six possible sums, and four of them give 1-spin projectors:

$$\begin{aligned}
\mathbf{P}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} + \bar{\mathbf{P}}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} &= \mathbf{P}_{\mathbf{s}_B} \\
\mathbf{P}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} + \mathbf{P}_{\mathbf{s}_A} \otimes \bar{\mathbf{P}}_{\mathbf{s}_B} &= \mathbf{P}_{\mathbf{s}_A} \\
\bar{\mathbf{P}}_{\mathbf{s}_A} \otimes \bar{\mathbf{P}}_{\mathbf{s}_B} + \bar{\mathbf{P}}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} &= \bar{\mathbf{P}}_{\mathbf{s}_A} \\
\bar{\mathbf{P}}_{\mathbf{s}_A} \otimes \bar{\mathbf{P}}_{\mathbf{s}_B} + \mathbf{P}_{\mathbf{s}_A} \otimes \bar{\mathbf{P}}_{\mathbf{s}_B} &= \bar{\mathbf{P}}_{\mathbf{s}_B}
\end{aligned} \tag{12}$$

whereas the other two give simple complementary 2-spin projectors that *correlate* the two spins (subsystems):

$$\begin{aligned}
\mathbf{P}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} + \bar{\mathbf{P}}_{\mathbf{s}_A} \otimes \bar{\mathbf{P}}_{\mathbf{s}_B} &= \frac{1}{2} (1 + \mathbf{s}_A \otimes \mathbf{s}_B) \\
\bar{\mathbf{P}}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} + \mathbf{P}_{\mathbf{s}_A} \otimes \bar{\mathbf{P}}_{\mathbf{s}_B} &= \frac{1}{2} (1 - \mathbf{s}_A \otimes \mathbf{s}_B).
\end{aligned} \tag{13}$$

4.5 Correlation and Entanglement

The spin-state density of a system in a *mixed* state is a linear combination of other spin-stgate densities. If it is a linear combination of decomposable spin-state densities

$$\varrho_{AB} = \sum_j w_j \varrho_A^j \otimes \varrho_B^j, \quad \sum_j w_j = 1, \tag{14}$$

it is called *disentangled* or *separable*. Although such a state carries no entanglement, it may still possess classical correlation, as we will see below. Examples of a separable spin-state densities are half of any of the simple correlated projectors above. In particular, the densities

$$\begin{aligned}
\varrho_{AB}^{(+)} &\equiv \frac{1}{2} \mathbf{P}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} + \frac{1}{2} \bar{\mathbf{P}}_{\mathbf{s}_A} \otimes \bar{\mathbf{P}}_{\mathbf{s}_B} = \frac{1}{4} (1 + \mathbf{s}_A \otimes \mathbf{s}_B) \\
\varrho_{AB}^{(-)} &\equiv \frac{1}{2} \bar{\mathbf{P}}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} + \frac{1}{2} \mathbf{P}_{\mathbf{s}_A} \otimes \bar{\mathbf{P}}_{\mathbf{s}_B} = \frac{1}{4} (1 - \mathbf{s}_A \otimes \mathbf{s}_B)
\end{aligned} \tag{15}$$

represent *classically* correlated systems. Primitive projectors can be formed from commuting products of simple correlated projectors, and these represent *entangled* systems (see below).

A *reduced spin-state density* for one subsystem of a composite system is found by summing ϱ over a complete sets of states of the other subsystems; this is equivalent to tracing the matrix representation of ϱ over the other subsystems. We can write

$$\varrho_{A\bar{B}} = n_B \langle \varrho_{AB} \rangle_{1 \otimes S} \simeq \text{tr}_B \{ \varrho_{AB} \} \tag{16}$$

for the reduced spin-state density of subsystem A after summing over the states of subsystem B . Here, n_B is the number of independent spin states of subsystem B and the subscript $1 \otimes S$ means that the scalar part of the second subsystem is taken. If subsystem B is a single qubit, $n_B = 2$. The result represents the spin-state density of subsystem A when the state of B is not determined. In the case of a product state $\varrho_{AB} = \varrho_A \otimes \varrho_B$,

$$\varrho_{A\bar{B}} = n_B \langle \varrho_{AB} \rangle_{1 \otimes S} = \varrho_A \otimes n_B \langle \varrho_B \rangle_S = \varrho_A .$$

To determine correlation, we compare the purity of the coupled system to that of the separate subsystems:

$$2^2 \left(\langle \varrho_{AB}^2 \rangle_{S \otimes S} - \langle (\varrho_{A\bar{B}} \otimes \varrho_{\bar{A}B})^2 \rangle_{S \otimes S} \right). \quad (17)$$

We could also use the von Neumann entropy for this purpose, but the measure above is generally somewhat simpler to evaluate. For classical correlation of the form (15), this measure gives a correlation of $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ whereas for a pure entangled state, the measure is $1 - \frac{1}{4} = \frac{3}{4}$. In fact, $\frac{1}{4}$ is the maximum correlation value by this measure for classically correlated states that have spin-state densities of the separable form (14), where the single-spin densities ϱ_A^j and ϱ_B^j are for pure orthogonal states:

$$\begin{aligned} \varrho_{AB}^2 &= \left(\sum_j w_j \varrho_A^j \otimes \varrho_B^j \right)^2 = \sum_j w_j^2 \varrho_A^j \otimes \varrho_B^j \\ 2^2 \langle \varrho_{AB}^2 \rangle_{S \otimes S} &= \sum_j w_j^2 \equiv x \leq 1 \\ (\varrho_{A\bar{B}} \otimes \varrho_{\bar{A}B})^2 &= \left(\sum_{jk} w_j w_k \varrho_A^j \otimes \varrho_B^k \right)^2 = \sum_{jk} w_j^2 w_k^2 \varrho_A^j \otimes \varrho_B^k \\ 2^2 \langle (\varrho_{A\bar{B}} \otimes \varrho_{\bar{A}B})^2 \rangle_{S \otimes S} &= \sum_{jk} w_j^2 w_k^2 \equiv x^2 \end{aligned}$$

The correlation measure (17) thus is $x^2 - x$ which, since $0 < x \leq 1$, has a maximum value of $\frac{1}{4}$ at $x = \frac{1}{2}$ for the two-qubit system. Any correlation beyond what is possible classically is a sign of *quantum correlation* and hence of *entanglement*.

Arbitrary states, including entangled (and hence inseparable) ones, can be expressed as linear combinations of product states:

$$\Psi_{AB} (P_3 \otimes P_3) = \sum_{j,k} c_{jk} \left(\Psi_A^j \otimes \Psi_B^k \right) (P_3 \otimes P_3) .$$

Each Ψ^j is a transformation amplitude and may include both unitary and projector factors. The unitary factors represent reversible transformations whereas the projectors are filters or, equivalently, measurements. The corresponding density operator is

$$\varrho_{AB} = \Psi_{AB} (\mathbf{P}_3 \otimes \mathbf{P}_3) \Psi_{AB}^\dagger = \sum_{j,k,j',k'} c_{jk} c_{j'k'}^* \left(\Psi_A^j \mathbf{P}_3 \Psi_A^{j'\dagger} \otimes \Psi_B^k \mathbf{P}_3 \Psi_B^{k'\dagger} \right).$$

This can be nonseparable and thus distinct from the form (14) if, for example, at least some of the terms in the above sum are not hermitian. The average component of spin A in the direction $\hat{\mathbf{a}}$ and, simultaneously, of spin B in the direction $\hat{\mathbf{b}}$, is $2^2 \langle \varrho_{AB} \hat{\mathbf{a}} \otimes \hat{\mathbf{b}} \rangle_{S \oplus S}$ where the subscript $S \oplus S$ indicates that the scalar part is to be taken in the spaces of both spin A and spin B. Similarly, the probability that spin A, upon appropriate measurement, is found in state $\mathbf{P}_{\mathbf{s}_A}$ and spin B is found in state $\mathbf{P}_{\mathbf{s}_B}$ is $2^2 \langle \varrho_{AB} \mathbf{P}_{\mathbf{s}_A} \otimes \mathbf{P}_{\mathbf{s}_B} \rangle_{S \oplus S}$.

One way that linear superpositions of product states arise is in reduction of representations of the rotation group. If we rotate Ψ_{AB} by an additional rotation with rotor R we get

$$\Psi_{AB} \rightarrow R \otimes R \Psi_{AB}$$

and the rotation factor $R \otimes R$ is a product representation of $SU(2)$. For example, suppose we rotate each spin by π about the axis $\hat{\mathbf{a}}$. Then $R = \exp(-i\hat{\mathbf{a}}\pi/2) = -i\hat{\mathbf{a}}$ and $R \otimes R = -\hat{\mathbf{a}} \otimes \hat{\mathbf{a}}$.

The rotation factor $R \otimes R$ can be reduced to a direct sum of spin-1 and spin-0 parts:

$$R \otimes R \simeq UR \otimes RU^\dagger = D^{(0)} \oplus D^{(1)},$$

where the elements of the 4×4 unitary matrix U are Clebsch-Gordan coefficients $\langle SM | s_1 m_1 s_2 m_2 \rangle$. This group reduction entangles decomposable states that contribute to the same irreducible representation. In particular, the $S, M = 1, 0$ and $S, M = 0, 0$ states are fully entangled combinations of spin-up/spin-down pairs. Consider the spin-0 part:

$$\psi_{AB}^{(0)} = \frac{1}{\sqrt{2}} (1 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes 1) (\mathbf{P}_3 \otimes \mathbf{P}_3)$$

with a spin density

$$\begin{aligned} \varrho_{AB}^{(0)} &= \psi_{AB}^{(0)} \psi_{AB}^{(0)\dagger} = \frac{1}{2} (1 - \mathbf{e}_1 \otimes \mathbf{e}_1) (\bar{\mathbf{P}}_3 \otimes \mathbf{P}_3 + \mathbf{P}_3 \otimes \bar{\mathbf{P}}_3) \\ &= \mathbf{P}_{-11} \mathbf{P}_{-33} = \frac{1}{4} (1 - \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3), \end{aligned}$$

where $P_{\pm jk} \equiv \frac{1}{2} (1 \pm \mathbf{e}_j \otimes \mathbf{e}_k)$ is a correlated 2-spin projector. Note that $\psi_{AB}^{(0)}$ for the singlet is antisymmetric under interchange, and $\varrho_{AB}^{(0)}$ is symmetric. One can also verify that $\psi_{AB}^{(0)}$ is invariant under rotations: $R \otimes R \psi_{AB}^{(0)} = \psi_{AB}^{(0)}$. It is easy to see that $\varrho_{AB}^{(0)}$ is idempotent and may be viewed as a 2-spin (or 2-qubit) projector. Thus, $\varrho_{AB}^{(0)}$ represents a pure state. Maximally entangled states are pure. However, mixed states can also be partially entangled. Note that if the entangled pure state with density $\varrho_{AB}^{(0)}$ is mixed with twice as much of an unpolarized mixture, we get a classical mixture of the same form but smaller polarization:

$$\begin{aligned} \frac{1}{3} \varrho_{AB}^{(0)} + \frac{2}{3} \left(\frac{1}{4} \right) &= \frac{1}{4} \left(1 - \frac{\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3}{3} \right) \\ &= \frac{1}{6} (P_{-11} + P_{-22} + P_{-33}) \end{aligned}$$

[compare (15)].

Consider the triplet state with $M_S = 0$. This is easily found within a normalization factor by starting with the $(S, M) = (1, 1)$ state

$$\begin{aligned} \psi_{AB}^{(1,1)} &= P_3 \otimes P_3 \\ \varrho_{AB}^{(1,1)} &= P_3 \otimes P_3 \end{aligned}$$

and applying the *lowering operator* [compare s_-] $S_- = \mathbf{e}_1 P_3 \otimes 1 + 1 \otimes \mathbf{e}_1 P_3$:

$$\begin{aligned} \psi_{AB}^{(1,0)} &= \frac{1}{\sqrt{2}} (1 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes 1) (P_3 \otimes P_3) \equiv \psi_{AB}^{(\mathbf{e}_3)} \\ \varrho_{AB}^{(1,0)} &= \psi_{AB}^{(1,0)} \psi_{AB}^{(1,0)\dagger} = \frac{1}{2} (1 + \mathbf{e}_1 \otimes \mathbf{e}_1) (\bar{P}_3 \otimes P_3 + P_3 \otimes \bar{P}_3) \\ &= P_{11} P_{-33} = \frac{1}{4} (1 + \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3) \\ &\equiv \varrho_{AB}^{(\mathbf{e}_3)}. \end{aligned}$$

Note $P_{11} = \frac{1}{2} (1 + \mathbf{e}_1 \otimes \mathbf{e}_1)$ and $P_{-11} = \frac{1}{2} (1 - \mathbf{e}_1 \otimes \mathbf{e}_1)$ are complementary simple 2-spin projectors, and the commuting products $P_{\pm 11} P_{\pm 33}$ are primitive projectors. A further application of the lowering operator gives

$$\begin{aligned} \psi_{AB}^{(1,-1)} &= (\mathbf{e}_1 \otimes \mathbf{e}_1) P_3 \otimes P_3 \\ \varrho_{AB}^{(1,-1)} &= \bar{P}_3 \otimes \bar{P}_3 \end{aligned}$$

The linear combinations of $(S, M) = (1, \pm 1)$ states,

$$\begin{aligned}\psi_{AB}^{(\mathbf{e}_1)} &= \frac{1}{\sqrt{2}} (1 - \mathbf{e}_1 \otimes \mathbf{e}_1) (\mathbf{P}_3 \otimes \mathbf{P}_3) = \sqrt{2} \mathbf{P}_{-11} (\mathbf{P}_3 \otimes \mathbf{P}_3) \\ \psi_{AB}^{(\mathbf{e}_2)} &= \frac{1}{\sqrt{2}} (1 + \mathbf{e}_1 \otimes \mathbf{e}_1) (\mathbf{P}_3 \otimes \mathbf{P}_3) = \sqrt{2} \mathbf{P}_{-22} (\mathbf{P}_3 \otimes \mathbf{P}_3)\end{aligned}$$

complete a basis $\left\{ \psi_{AB}^{(0)}, \psi_{AB}^{(\mathbf{e}_1)}, \psi_{AB}^{(\mathbf{e}_2)}, \psi_{AB}^{(\mathbf{e}_3)} \right\}$ of fully entangled 2-spin states. The spin-state densities are the primitive projectors

$$\begin{aligned}\varrho_{AB}^{(0)} &= \mathbf{P}_{-11} \mathbf{P}_{-33} = \mathbf{P}_{-11} \mathbf{P}_{-22} \mathbf{P}_{-33} \\ \varrho_{AB}^{(\mathbf{e}_1)} &= \mathbf{P}_{-11} \mathbf{P}_{33} = \mathbf{P}_{-11} \mathbf{P}_{22} \mathbf{P}_{33} = \frac{1}{4} (1 - \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \\ \varrho_{AB}^{(\mathbf{e}_2)} &= \mathbf{P}_{11} \mathbf{P}_{33} = \mathbf{P}_{11} \mathbf{P}_{-22} \mathbf{P}_{33} = \frac{1}{4} (1 + \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \\ \varrho_{AB}^{(\mathbf{e}_3)} &= \mathbf{P}_{11} \mathbf{P}_{-33} = \mathbf{P}_{11} \mathbf{P}_{22} \mathbf{P}_{-33} = \frac{1}{4} (1 + \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3) .\end{aligned}$$

We have used the idempotency of the projectors \mathbf{P}_{jk} together with the pacman property to relate, for example,

$$\mathbf{P}_{11} \mathbf{P}_{33} = \mathbf{P}_{11} \mathbf{P}_{11} \mathbf{P}_{33} = \mathbf{P}_{11} \frac{1}{2} (1 + \mathbf{e}_1 \mathbf{e}_3 \otimes \mathbf{e}_1 \mathbf{e}_3) \mathbf{P}_{33} = \mathbf{P}_{11} \mathbf{P}_{-22} \mathbf{P}_{33}$$

since $\mathbf{e}_1 \mathbf{e}_3 = -i \mathbf{e}_2$.

Any state ψ_{AB} is a linear combination of the fully entangled basis states as well as a linear combination of the product (separable) state basis

$$\{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \} = \{ 1, 1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes 1, \mathbf{e}_1 \otimes \mathbf{e}_1 \} (\mathbf{P}_3 \otimes \mathbf{P}_3) .$$

The entangled states can also be found by projection from a generic state. For example,

$$\begin{aligned}\mathbf{P}_{-11} \mathbf{P}_{-33} (a + b \mathbf{1} \otimes \mathbf{e}_1 + c \mathbf{e}_1 \otimes 1 + d \mathbf{e}_1 \otimes \mathbf{e}_1) (\mathbf{P}_3 \otimes \mathbf{P}_3) \\ = \mathbf{P}_{-11} (b \mathbf{1} \otimes \mathbf{e}_1 + c \mathbf{e}_1 \otimes 1) (\mathbf{P}_3 \otimes \mathbf{P}_3) \\ = \frac{(b - c)}{2} (1 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes 1) (\mathbf{P}_3 \otimes \mathbf{P}_3) .\end{aligned}$$

The spin-state densities of the fully entangled states can thus be expressed as either the singlet

$$\varrho_{AB}^{(0)} = \frac{1}{4} (1 - \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3) \quad (18)$$

or as the triplet

$$\varrho_{AB}^{(\mathbf{n})} = \frac{1}{4} (1 + \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) - \frac{1}{2} \mathbf{n} \otimes \mathbf{n} \quad (19)$$

in terms of the unit spatial vector \mathbf{n} . Under rotations of the triplet state, \mathbf{n} rotates as a vector:

$$(R \otimes R) \varrho_{AB}^{(\mathbf{n})} (R^\dagger \otimes R^\dagger) = \varrho_{AB}^{(R\mathbf{n}R^\dagger)}.$$

In particular, if R is a rotor that takes \mathbf{e}_3 into \mathbf{n} , then

$$\varrho_{AB}^{(\mathbf{n})} = (R \otimes R) \varrho_{AB}^{(\mathbf{e}_3)} (R^\dagger \otimes R^\dagger)$$

and the eigenspinor whose pure spin-state density is $\varrho_{AB}^{(\mathbf{n})}$ has the form

$$\psi_{AB}^{(\mathbf{n})} = R \otimes R \psi_{AB}^{(\mathbf{e}_3)}.$$

It is easy to see that $\varrho_{AB}^{(\mathbf{n})}$, like $\varrho_{AB}^{(\mathbf{e}_3)} = P_{11}P_{22}$ is idempotent and because each reduced spin-state density (16) is unpolarized,

$$\varrho_{AB}^{(\mathbf{n})} = \varrho_{AB}^{(\mathbf{n})} = \frac{1}{2},$$

it has the maximum correlation measure (17) of $\frac{3}{4}$.

The densities (18) and (19) have the same form as those for separable states (14), but the states are not separable, because the terms being summed are not state densities. Instead, $\varrho_{AB}^{(\mathbf{n})}$ and $\varrho_{AB}^{(0)}$ are pure entangled states. However, an equal mixture of $\varrho_{AB}^{(0)}$ and $\varrho_{AB}^{(\mathbf{n})}$ is separable:

$$\varrho'_{AB} = \frac{1}{2} [\varrho_{AB}^{(\mathbf{n})} + \varrho_{AB}^{(0)}] = \frac{1}{2} P_{-\mathbf{n}\mathbf{n}} = \frac{1}{4} (1 - \mathbf{n} \otimes \mathbf{n}) = \frac{1}{2} (P_{\mathbf{n}} \otimes \bar{P}_{\mathbf{n}} + \bar{P}_{\mathbf{n}} \otimes P_{\mathbf{n}}).$$

These are *separable* states, that is mixtures of decomposable (uncorrelated) states. Through the mixture, the two spins are *classically correlated*, but not entangled; the correlation is purely classical. These mixed states are partially polarized, since

$$\varrho_{AB}^{\prime 2} = \frac{1}{2} \varrho'_{AB}$$

whereas for the unpolarized case

$$\varrho_{AB} = \frac{1}{4} (1 \otimes 1), \quad \varrho_{AB}^2 = \frac{1}{4} \varrho_{AB},$$

and the reduced matrices are fully unpolarized, for example,

$$\varrho'_{\bar{A}B} = 2 \langle \varrho'_{AB} \rangle_{S \otimes 1} = \frac{1}{2} .$$

More generally, the state densities equal to the simple projectors

$$\begin{aligned} \frac{1}{2} P_{-\mathbf{n}\mathbf{m}} &= \frac{1}{4} (1 - \mathbf{n} \otimes \mathbf{m}) = \frac{1}{2} (P_{\mathbf{n}} \otimes \bar{P}_{\mathbf{m}} + \bar{P}_{\mathbf{n}} \otimes P_{\mathbf{m}}) \\ \frac{1}{2} P_{\mathbf{n}\mathbf{m}} &= \frac{1}{4} (1 + \mathbf{n} \otimes \mathbf{m}) = \frac{1}{2} (P_{\mathbf{n}} \otimes P_{\mathbf{m}} + \bar{P}_{\mathbf{n}} \otimes \bar{P}_{\mathbf{m}}) \end{aligned}$$

are separably correlated. Products of simple correlated projectors are needed to form the spin density of any entangled state.

The most general fully entangled state seems to be of the form

$$\begin{aligned} \psi_{AB} &= R_A \otimes R_B \psi_{AB}^{(0)} \\ \varrho_{AB} &= \psi_{AB} \psi_{AB}^\dagger = R_A \otimes R_B P_{-11} P_{-33} R_A^\dagger \otimes R_B^\dagger , \end{aligned}$$

where R_A and R_B are independent rotors. Unlike $\varrho_{AB}^{(0)}$ and $\varrho_{AB}^{(\mathbf{n})}$, this ϱ_{AB} is not generally symmetric under interchange of the two qubits.

4.6 Total Spin and Nonlocal Projectors

We can project a fully entangled state out of a separable one (15) by applying a nonlocal spin projectors. Consider the singlet-state projector for the two-spin system:

$$P_{S=0} = 1 - \frac{1}{2} S^2,$$

where

$$S^2 = \sum_j S_j^2 = \frac{1}{4} \sum_{j=1}^3 (\mathbf{e}_j \otimes 1 + 1 \otimes \mathbf{e}_j)^2 = \frac{1}{2} \sum_j (1 + \mathbf{e}_j \otimes \mathbf{e}_j)$$

is the square of the total spin operator. Substitution of S^2 into $P_{S=0}$ gives directly the isotropic form

$$\begin{aligned} P_{S=0} &= 1 - \frac{1}{4} [3 + \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3] \\ &= \frac{1}{4} [1 - (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)] \\ &= P_{-11} P_{-33} . \end{aligned}$$

Applying $P_{S=0}$ to $\varrho_{AB}^{(-)}$ (15) with applications of the pacwoman property,

$$P_{S=0} (P_3 \otimes \bar{P}_3 + \bar{P}_3 \otimes P_3) = \frac{1}{2} (1 - \mathbf{e}_1 \otimes \mathbf{e}_1) (P_3 \otimes \bar{P}_3 + \bar{P}_3 \otimes P_3)$$

we find

$$P_{S=0} \varrho_{AB}^{(-)} P_{S=0} = \frac{1}{2} \varrho_{AB}^{(0)}$$

which is just the singlet contribution to ϱ' (15). Similarly, the spin-1 projector:

$$\begin{aligned} P_{S=1} &= \frac{1}{2} S^2 = 1 - P_{S=0} = P_{11} P_{-33} + P_{11} P_{33} + P_{-11} P_{33} \\ &= P_{11} P_{-33} + P_{33} \end{aligned}$$

is complementary to the spin-0 projector and projects onto the triplet state:

$$P_{S=1} \varrho_{AB}^{(-)} P_{S=1} = \frac{1}{2} \varrho_{AB}^{(\mathbf{e}_3)},$$

However, since $P_{S=1}$ is not primitive, whether or not it produces an entangled state depends on the target state.

5 Conclusions

APS provides a natural and efficient representation of qubits as spin- $\frac{1}{2}$ systems. Multiple qubit systems are represented by tensor products of the minimal ideal of APS with the projector P_3 . Operations are all represented algebraically without matrices or tensors. Entanglement can be defined as a quantum correlation of subsystems. A higher state purity or polarization in the complete system than in the tensor product of its subsystems is what defines correlation, but it is not always sufficient since it also holds for some classically correlated states. However, the purity of classically correlated states is limited, and we can define entanglement generally as quantum correlation, which exists for correlated states whose purity exceeds the classical limit. Simple two-spin correlated projectors are useful for representing correlated states. The state densities of fully entangled 2-spin systems are primitive projectors, which can be expressed as the products of commuting simple projectors. The definitions of correlation and entanglement are readily extended to systems of larger numbers of qubits.

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