Understanding Electromagnetic Radiation from an Accelerated Charge

William E. Baylis

Physics Department, University of Windsor,

Windsor, Ontario, Canada N9B 3P4

Abstract

In spite of its importance and solid theoretical underpinnings, the formation of electromagnetic (EM) radiation by accelerating charges is still a source of wonder if not bewilderment. The conceptual understanding of how radiation is generated is the subject of this paper.

Maxwell's equations, together with causality, dictate that charges carry an electromagnetic field whose effects propagate in free space at the speed of light in straight lines. The field at any point thus depends on the charge and its motions at an earlier "retarded" time. Part of the field is just the Coulomb field boosted from rest to the velocity of the charge at the retarded time. The electric part of the boosted Coulomb field of a positive charge always points away from the *inertial image* of the charge, that is the instantaneous position the charge would have if it continued with the velocity it had at the retarded time. For a charge moving at constant velocity, the inertial image is the actual position of the charge, there is no radiation, and the flux lines of the electric field are straight and pass through the instantaneous position of the charge. For an accelerated charge, the boosted Coulomb field lines are broken. The radiation field must be added to the transformed Coulomb field to connect the lines and therefore to satisy Maxwell's equations. Geometric methods and relativistic symmetries help illuminate these processes. They also suggest and help prove a simple algorithm for computing the field lines of a charge in arbitrary rectilinear motion. Uniform acceleration is considered as a special case and used to determine the field of a static charge in a gravitational field.

I. INTRODUCTION

The purpose of this paper is to describe how radiation arises from accelerated charges. Many treatments discuss the generation of a magnetic field **B** by changes in the electric field **E** and vice versa to explain the propagation of fields at the speed of light, and some describe the radiation as the field "shaken loose" or broken away from an accelerating charge.[1] The mathematics is well known, showing how Maxwell's equations, together with causal boundary conditions, predict radiation, but the conceptual understanding still seems somewhat unsatisfying. Any disturbance or information in an electromagnetic field, not just radiation, propagates in free space at the speed of light. What is special about radiation fields? Why is radiation necessary when the charge is accelerating, and how is it shaken loose from the charge?

The approach here emphasizes relativistic (covariant) symmetries, especially the unity of the electric and magnetic fields as aspects of a single field **F**.[2] We introduce the concept of the "inertial image" of the charge to help show that the transformed Coulomb field lines break when the charge is accelerated, and that the radiation field is necessary to rejoin them. A simple "virtual photon" algorithm for computing the field lines of a charge in arbitrary rectilinear motion lets students write their own code to compute field lines. A final part discusses the relation of a uniformly accelerated charge to a charge at rest in a gravitational field. Mathematical details, which avoid tensor indices and matrices by using the algebra of physical space (APS),[3, 4] can be found in the Appendix.

II. COVARIANT FIELD F

Electric and magnetic fields of a propagating wave are parts of a *single* covariant field **F**, which describes a plane at each point in spacetime. (The covariance of **F** refers to the linear way the field **F** seen by one inertial observer is related to that seen by another.) The field of a point charge (its Liénard-Wiechert field) has both boosted Coulomb ($\mathbf{F}_c$) and radiative ($\mathbf{F}_r$) parts[2, 5] [SI units with times measured in distance units ($c = 1$) are used]

$$\mathbf{F} = \mathbf{F}_c + \mathbf{F}_r = \mathbf{E} + i\mathbf{B} \ , \tag{1}$$

where the electric field $\mathbf{E} = \mathbf{E}_c + \mathbf{E}_r$ is the *timelike* component of the planes **F** and $i\mathbf{B}$ is their *spatial* component, the normal vector (dual) of which is the usual magnetic field vector $\mathbf{B}$. Much of the discussion below centers on the *electric-field lines*. These are the intersections of the spacetime planes $\mathbf{F}$ with the observer's physical space, that is with the hypersurface orthogonal to the observer's time axis, which is identified from another frame as the observer's proper velocity. Different inertial observers see different intersections of $\mathbf{F}$ and thus different electric fields.

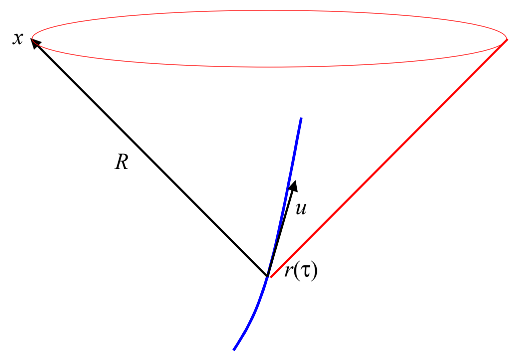

FIG. 1: x is the field point and $r(\tau)$ is the world line of the charge.

The charge follows a path or world line $r(\tau)$ in spacetime parametrized by its proper time τ. The field $\mathbf{F}(x)$ at spacetime position x depends only on the charge and its motion (on r and its two derivatives with respect to τ, $u = \dot{r}$ and $\dot{u}$) at one spacetime point $r(\tau_x)$ on its world line. The position $r(\tau_x)$ is the position of the charge as seen by an observer at x. The *retarded position* $r(\tau_x)$ is the one position on the world line whose forward light cone contains x. By causality, the relative spacetime vector $R = R^0\mathbf{e}_0 + \mathbf{R} \equiv x - r(\tau_x)$ has a positive time component $R^0 = x^0 - r^0 > 0$ on the time axis $\mathbf{e}_0 \equiv 1$, and by Maxwell's equations it is *null*: $R^0 = |\mathbf{R}|$. The proper time τ_x of the charge that fulfills this *light-cone condition* is called the *retarded time*. It is obviously a function of the field position x (τ_x is a scalar field): $\tau_x = \tau(x)$. Mathematical details are given in the Appendix.

The boosted Coulomb field $\mathbf{F}_c$ has an electric part $\mathbf{E}_c$ pointing radially away from what we call the *inertial image* of the charge. The inertial image, located at $\mathbf{r}(\tau_x) + \mathbf{v}R^0$, is the *instantaneous position* the charge would have if it continued at constant velocity from its retarded position. Thus, a charge moving *at constant velocity* $\mathbf{v}$ is located at its inertial image. Whereas its retarded position is $\mathbf{r}(\tau_x) = \mathbf{r}(0) + \mathbf{v}r^0$, its inertial image is at the instantaneous position

$$\mathbf{r}(\tau_x) + \mathbf{v}R^0 = \mathbf{r}(0) + \mathbf{v}r^0 + \mathbf{v}\left(x^0 - r^0\right) \tag{2}$$

$$= \mathbf{r}(0) + \mathbf{v}x^0, \tag{3}$$

which lies ahead a distance $\mathbf{v}\left(x^0 - r^0\right) = \mathbf{v}R^0$ and depends only on the coordinate time x^0 of the observer, not on her spatial position $\mathbf{x}$.

The electric part $\mathbf{E}_c$ of the boosted Coulomb field $\mathbf{E}_c$ at x, consists of straight lines from the inertial image toward $\mathbf{x}$, that is in the direction

$$\mathbf{x} - \left[\mathbf{r}(\tau) + \mathbf{v}R^0\right] = \mathbf{R} - \mathbf{v}R^0. \tag{4}$$

Of course, the electric field of a charge moving at constant velocity *must* have this form. It is related by a linear transformation (the boost) to the field of a static charge. We know that the field of a static charge consists of straight lines from the charge, and a linear transformation does not change this. The strength of the field is proportional to the density of field lines, and since these are radial, $|\mathbf{E}_c|$ falls off as $\mathbf{R}^{-2}$.

The boost does add something to the electric field: by sweeping the electric field lines along $\mathbf{v}$, it generates *spatial planes* containing both $\mathbf{E}$ and $\mathbf{v}$, and these represent the *magnetic field*. The field $\mathbf{B}$ is normal to these planes and thus circles around $\mathbf{v}$, as seen in Figure 2. There is no radiation field from a charge moving with constant velocity.

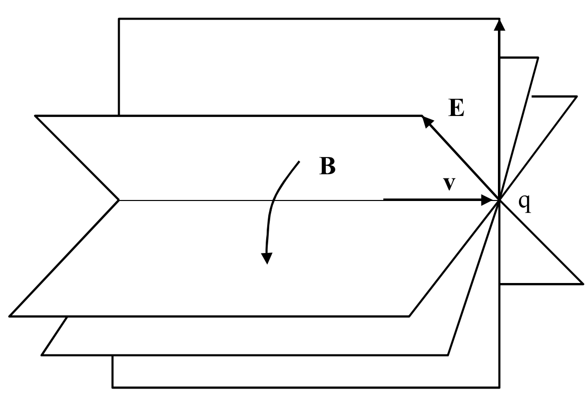

FIG. 2: Electric and magnetic fields of a moving charge q.

The electric field lines of a charge moving at constant velocity thus appear to be rigidly fixed to the charge. Physically, however, the field information propagates at the speed of light in straight lines from the *retarded position* of the charge.

III. RADIATION

What is different when the charge is accelerated? The velocity of the inertial image is different at different retarded times. At a given instant x^0, the inertial image as seen from different positions $\mathbf{x}$ is at different positions. The field $\mathbf{E}_c$ at x is always directed from the inertial image to $\mathbf{x}$, but the image position now changes with $\mathbf{x}$. As a result, the field lines for $\mathbf{E}_c$ can no longer be straight. In fact, it is worse: they cannot even be connected! The acceleration *breaks* the Coulomb field lines.

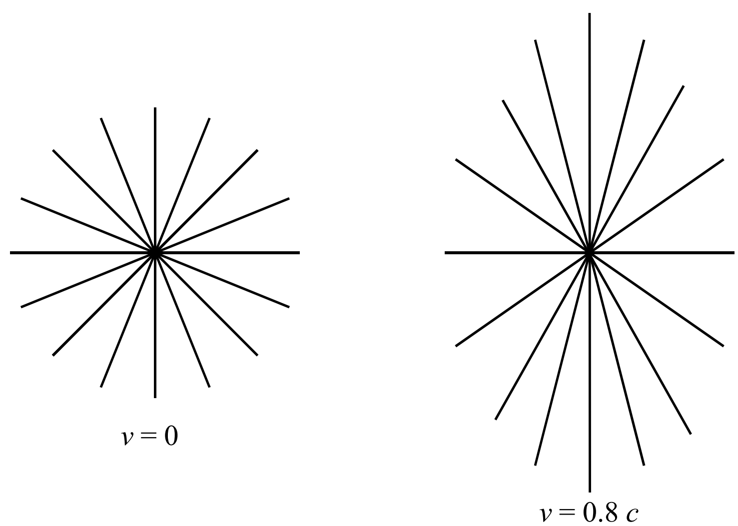

FIG. 3: The electric field lines of a point charge.

To see this, consider a charge that has been at rest at the origin for a long time, but at time $r^0 = 0$ suddenly receives a sharp impulse that accelerates it rapidly to high velocity, say $\mathbf{v} = 0.8\,\mathbf{e}_1$ in units of the speed of light, where $\mathbf{e}_1$ is a fixed spatial unit vector. The field lines for the moving charge bunch up toward the plane perpendicular to $\mathbf{v}$, as shown in Figure 3. The Coulomb field lines perpendicular to $\mathbf{v}$ maintain their direction under the boost, and by symmetry one expects them to be connected together, but they point to different positions of the inertial image, depending on whether the retarded time is before or after the impulse.

At observation time x^0, the inertial image seen at points $\mathbf{x}$ with $|\mathbf{x}| > x^0$ is at the origin, but at points $\mathbf{x}$ closer to the origin than about x^0, the image is seen to be approximately at $\mathbf{v}x^0$. In the transition region, the image position lies between the origin and $\mathbf{v}x^0$. There is no way that a single field line can remain perpendicular to $\mathbf{v}$ and also point to the origin at large distances and to $\mathbf{v}x^0$ at smaller ones. The Coulomb field line has been broken by the acceleration, as shown in Figure 4.

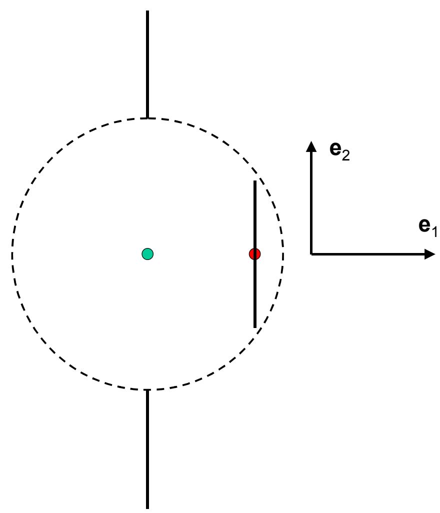

FIG. 4: A charge at rest at the origin for a long time receives a sharp impulse at $t = 0$ that accelerates it to $\mathbf{v} = 0.8\mathbf{e}_1$. The Coulomb field line along $\mathbf{e}_2$ is broken. It will be connected by the radiation field. The small circles indicate the positions of the inertial image inside and outside the dashed spherical region.

If the $\mathbf{E}_c$ lines are broken, then $\mathbf{F}_c$ no longer satisfies Maxwell's equations. In particular, $\nabla \cdot \mathbf{E}_c \neq 0$ in the source-free region. However, the total field $\mathbf{F}$ still obeys Maxwell's equations. It is evidently the *radiation part* $\mathbf{F}_r$ that *connects the broken Coulomb lines.*

The radiation part $\mathbf{F}_r = \left(1 + \hat{\mathbf{R}}\right) \mathbf{E}_r$ is called a *null flag* with flagpole $1+\hat{\mathbf{R}}$. It is linear in the retarded acceleration $\dot{u}$ and thus arises only during periods of acceleration. The electric field $\mathbf{E}_r$ is perpendicular to $\hat{\mathbf{R}}$ and the magnetic field $\mathbf{B}_r$ is normal to the plane $\hat{\mathbf{R}}\mathbf{E}_r$ swept out by $\mathbf{E}_r$ as the wave propagates along $\hat{\mathbf{R}}$. Thus the electric and magnetic fields are different aspects of a single radiation field $\mathbf{F}_r$, much as in acoustics, the velocity and pressure waves are different aspects of a single longitudinal sound wave. In each case, changes in the one aspect can be seen as driving the oscillation of the other. The field $\mathbf{E}_r$ is perpendicular to $\mathbf{B}_r$ and to $\mathbf{R} = \mathbf{x} - \mathbf{r}(\tau_x)$, and it is able to join the broken pieces of the Coulomb field lines. All the Coulomb lines are broken except those in the forward and backward direction, where by symmetry they remain connected. We therefore expect the radiation to vanish in directions collinear with the acceleration. Because the lengths of the radiation segments needed to connect the broken Coulomb lines grow as $|\mathbf{R}|$, the line density (and hence the field strength) of the radiation falls off as $|\mathbf{R}|^{-1}$.

The split $\mathbf{F} = \mathbf{F}_c + \mathbf{F}_r$ is covariant: the same for every inertial observer. However, the split $\mathbf{F} = \mathbf{E} + i\mathbf{B}$ is observer-dependent: for example, an inertial observer instantaneously moving with the charge will detect no magnetic field. Radiation occurs as acceleration "shakes" the charge away from its inertial image and thereby breaks the Coulomb field. $\mathbf{F}_r$ rejoins the broken $\mathbf{F}_c$. Neither $\mathbf{F}_c$ nor $\mathbf{F}_r$ generally satisfies Maxwell's equation, but the sum $\mathbf{F}$ does, and consequently connected $\mathbf{E}$ field lines are guaranteed only for $\mathbf{F}$.

IV. VIRTUAL-PHOTON METHOD

The basic idea that the electromagnetic fields propagate in straight lines at the speed of light, plus the fact that the vector potential depends only on the position and proper velocity of the charge at the retarded time, leads to a very simple method to compute field lines for a charge in rectilinear motion.[2] We have implemented the method in simple Maple algorithms. We call it the *virtual-photon method.*

We envision virtual "photon streams" that are emitted continuously and isotropically from the charge *in its rest frame* (*i.e.,* in the inertial frame instantaneously commoving with the charge). The virtual photons move along straight rays on the light cones of the charge. Although the virtual photons are radiated isotropically, it is useful to think of them, in the nature of field lines, as being emitted along fixed rays in the instantaneous rest frame of the charge. We thus imagine the virtual photons as concentrated in a number e/ε_0 of rays, evenly distributed around the charge in its rest frame. Although each virtual photon moves in a fixed spatial direction at the speed of light, the *stream* comprises photons emitted at different times from the moving source. If the source is jerked, the stream develops a kink, much as would a *stream of water from a hose*, even though each water molecule follows a ballistic trajectory. The virtual photon streams are the field lines.

To understand the method, consider first the virtual photons from a charge in *uniform* *motion*, as computed in a simple Maple worksheet. In the worksheet, the charge moves along the x axis at the speed v. It crosses the origin at the retarded time tr = 0. The field is observed at t = 10. Virtual photons are shown for emissions every time unit at rest-frame angles that are integer multiples of 180/N degrees. The angles are Lorentz-transformed to the lab frame. Once emitted, the virtual photons continue to travel at the speed of light in a fixed direction. The photons emitted at different times but in a given rest-frame direction line up to form the field lines. In this short program, (x,y) is the position of the virtual photon.

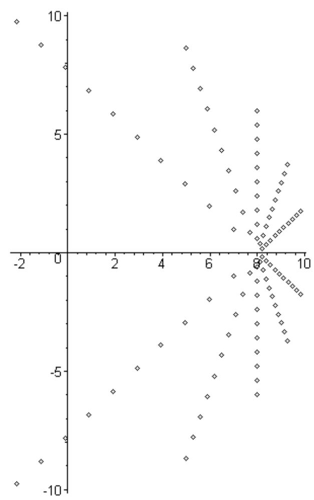

FIG. 5: Virtual photon plot for charge moving at $\mathbf{v} = 0.8\mathbf{e}_1$.

```
> restart; t:=10: N:=6: v:=0.8: points:=NULL:
> for n from 1 to N-1 do
>       cs:=cos(Pi*n/N): sn:=sin(Pi*n/N):
>       for tr from 0 to t-1 do
>             x:=v*tr+(t - tr)*(v+cs)/(1+v*cs):
>             y:=(t - tr)*sqrt(1 - v^2)*sn/(1+v*cs):
>             points:=points,[x,y],[x,-y]
>       od:
> od:
> plot([points],style=point,scaling=constrained);
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That is all there is to it. The result is shown in Figure 5. Note that the virtual photons are *not* moving directly away from the inertial image of the charge (which in this case is the

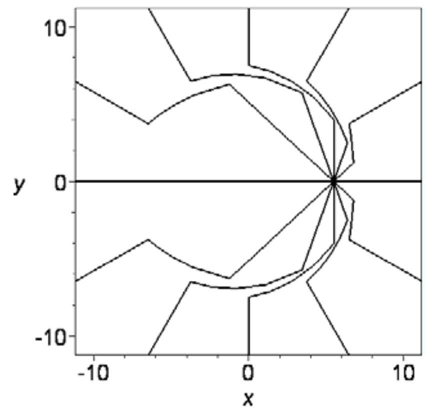

FIG. 6: The electric field lines of a briefly accelerated charge.

instantaneous position of the charge). Rather, they are moving at the speed of light away from the *retarded position* of the charge. For example, the virtual photons on the vertical lines are moving at angle θ to $\mathbf{v}$ such that $|\mathbf{v}| = \cos\theta$, that is, so that their horizontal motion keeps pace with the charge.

The method works just as well if the charge is arbitrarily accelerating along the $\mathbf{e}_1$ axis. It is also easily extended to make animations of the fields. Figure 6 shows one frame from the field lines of a charge initially at rest at the origin that receives a sudden impulse. The radiation field in this case is concentrated on the spherical shell whose retarded time corresponds to the brief period of acceleration.

In spacetime terms, as the charge progresses along its world line, the rays sweep out continuous spacetime *sheets* that cut the sequence of light cones of the charge. As we will see below, the Liénard-Wiechert field of the charge is the tangent plane to the sheet at the field position and it is the slice of such sheets at an instant in time that gives the electric field lines.

V. FIELD LINES OF UNIFORMLY ACCELERATED CHARGE

With the help of Clifford's geometric algebra of physical space (APS), we can easily find the electromagnetic field $\mathbf{F}$ for a charge in uniform acceleration. Let the charge move on the spatial axis $\mathbf{e}_2$ and let $\tau = 0$ be the proper time when it is instantaneously at rest. If we express length in units of a^{-1}, where a is the constant acceleration as seen in the rest frame (and as before, $c = 1$), the proper velocity of the charge can be written $u = \exp(\tau \mathbf{e}_2)$. We find (see Appendix for details of the formulation)

$$\dot{u} = r = \mathbf{e}_2 u \tag{5}$$

$$\bar{\dot{u}} u = -\mathbf{e}_2 \tag{6}$$

and the Liénard-Wiechert field reduces to the surprisingly simple expression[2]

$$\mathbf{F} = -\frac{Ke}{2 \langle x\bar{u} \rangle_S^3} (\mathbf{e}_2 + x\mathbf{e}_2\bar{x}) \ , \tag{7}$$

with $K = (4\pi\varepsilon_0)^{-1}$. This is real at $x^0 = t = 0$, the instant at which the charge comes to rest, and thus the magnetic field vanishes then. Both $\mathbf{B}_c$ and $\mathbf{B}_r$ are nonzero, but they cancel at $t = 0$. The field $\mathbf{F}$ is thus purely electric at $x^0 = 0$, and its tangent vectors are shown in the field plot in Figure 7.

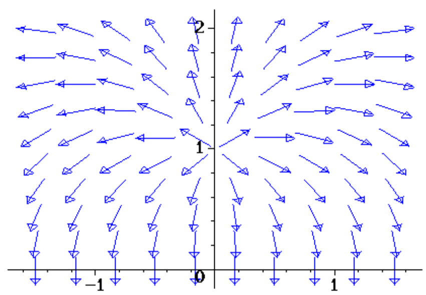

FIG. 7: Tangent vectors to $\mathbf{E}$ for uniformly accelerated charge.

The geometry and time dependence is seen in a spacetime diagram of the field sheets swept out by the virtual photon streams in time (see Figure 8). The intersection of the field sheets with the spatial hyperplane at any instant, shown in the figure as the top edge of the sheets at the instant the charge is at rest, give the electric field lines.

We see once again that for an accelerating charge, neither $\mathbf{F}_c$ nor $\mathbf{F}_r$ gives continuous field lines. The radiation field is just the bit needed so that the *sum* $\mathbf{F} = \mathbf{F}_c + \mathbf{F}_r$ satisfies Maxwell's equations.

A. Principle of Equivalence

According to Einstein's principle of equivalence,[6] a system in uniform acceleration should be indistinguishable locally from a system in a uniform gravitational field. Thus

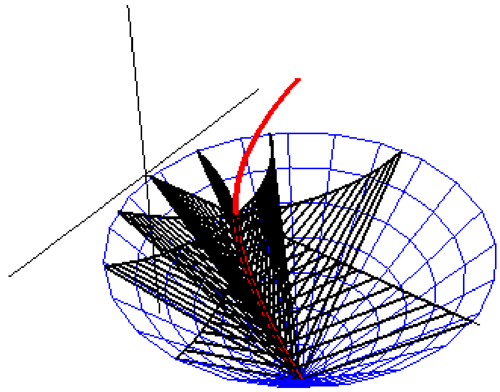

FIG. 8: Field sheets of an accelerating charge. Time is measured on the vertical axis, and the thick vertical arc is the world line of the charge.

we can interpret the field lines of the uniformly accelerated charge as electric field lines of a static charge in a gravitational field. It is appropriate that the magnetic field vanishes at the instant the charge turns around since a magnetic field for a static charge, even one in a gravitational field, would conflict with time-reversal symmetry.

But what about radiation? Does the static charge in the gravitational field radiate? Of course, this is an old problem, one still subject to controversy.[7, 8] The work presented here does not provide a complete analysis and therefore will not fully resolve the question, but it does offer some insight. For the case of the accelerating charge, a radiation field was required in order to connect the Coulomb field lines broken by the acceleration. In a gravitational field, that is no longer necessary: the Coulomb lines are not broken, only "bent" by gravity. Also note that the covariant split $\mathbf{F} = \mathbf{F}_c + \mathbf{F}_r$, which is the same for all inertial observers, is *different* for accelerated observers. In particular, acceleration and hence $\mathbf{F}_r$ disappears for a co-accelerating observer.

Thus, the *field lines* are the same; it's the $\mathbf{F}_c/\mathbf{F}_r$ split and thus the *interpretation* that changes when there is a gravitational field instead of an accelerated charge.

B. Extension: Electric Dipole in Gravitational Field

Here is a problem that nicely demonstrates the internal self-consistency of physics. Students can easily work out the answer, given the simple analytical form of the field (7).

An electric dipole in a gravitational field experiences a weak upward force that arises from the curvature of the field lines. We can use the principle of equivalence and the field lines of a uniformly accelerated charge to demonstrate the effect. Show that at the instant the charge e is at rest at $r = \mathbf{e}_2$, namely $x^0 = 0$, the electromagnetic field (7) at $\mathbf{x}$ is

$$\mathbf{F} = \frac{4Ke\left[2\mathbf{x}\,\mathbf{x}\cdot\mathbf{e}_2 - (1+\mathbf{x}^2)\,\mathbf{e}_2\right]}{\left|(1+\mathbf{x}^2)^2 + (2\mathbf{x}\cdot\mathbf{e}_2)^2\right|^{3/2}}\,. \tag{8}$$

Note that $\mathbf{F}$ is a pure electric field at this instant, even though separately the Coulomb and radiation fields both have nonvanishing magnetic components. Find the force on a negative charge $-e$ located at $\mathbf{x} = \mathbf{e}_2 + \xi\mathbf{e}_1$. Assume that the world line of the negative charge is like that of the positive charge, but displaced by a small distance ξ along the $\mathbf{e}_1$ axis. Use symmetry to find the force on the positive charge arising from the field lines of the negative one. Show that in the limit of small separations, $\xi \ll 1$, the dipole experiences an upward electric force given by $\Delta m\mathbf{a}$, where $\mathbf{a}$ is the uniform acceleration of the charges (the negative of the gravitational acceleration) and Δm is the amount by which the mass of the dipole decreases due to the attractive interaction of the opposite charges:

$$\Delta m = \frac{Ke^2}{\xi}\,. \tag{9}$$

Acknowledgments

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[1] See, for example, H. C. Ohanian, *Classical Electrodynamics (*Allyn and Bacon, Newton, Massachusetts, 1988), and D. J. Griffiths, *Introduction to Electrodynamics,* 3rd edition (Prentice Hall, Upper Saddle River, NJ, 1999).

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APPENDIX A: SOME MATH DETAILS

1. Liénard-Wiechert potential

The Liénard-Wiechert potential is simple enough, but the index-free covariant formalism[2] of APS helps us understand it. The potential can be derived by a boost from the rest-frame Coulomb potential:

$$\Phi_{\text{rest}} = \frac{Ke}{\langle R_{\text{rest}} \rangle_S}. \tag{A1}$$

Here,

$$R(\tau) = x - r(\tau) \equiv R^0 + \mathbf{R}, \tag{A2}$$

is the difference paravector between the field position x and the world line of the charge at the retarded proper time τ. Paravectors are elements of a four-dimensional linear space that models spacetime. They can be written as a scalar (the time component) plus a vector. The denominator $\langle R_{\text{rest}} \rangle_S$ is the time component (scalar part) of the difference (A2) in the commoving inertial frame. It is a Lorentz invariant simply because it is measured in the rest frame of the charge at the retarded time, but we make it manifestly invariant by setting $\langle R_{\text{rest}} \rangle_S = \langle R\bar{u} \rangle_S$, where $u = \gamma(1 + \mathbf{v})$ is the proper velocity of the charge at τ, which together with $R = R(\tau)$ can be in any inertial frame. The bar on $\bar{u}$ indicates the Clifford conjugate, which changes the sign of the spatial components: $\bar{u} = \gamma(1 - \mathbf{v})$. The scalar term $\langle R\bar{u}\rangle_S = \left(R\bar{u} + u\bar{R}\right)/2$ is the scalar product of the spacetime vectors R and u, and such scalar products are invariant under Lorentz transformations.

How do we get a covariant expression out of Φ? We only need to boost it to the proper velocity of the charge at the retarded τ to get the paravector potential for a charge in general motion:

$$A(x) = \Lambda\Phi_{\text{rest}}\Lambda^{\dagger} = \frac{Keu}{\langle R\bar{u}\rangle_S}\ , \tag{A3}$$

where Λ is the Lorentz rotor that induces the boost from rest to proper velocity $u = \Lambda\Lambda^{\dagger}$, and the dagger indicates Hermitian conjugation (algebraic reversal in APS). It is that simple. Note that $A(x)$ is covariant and depends only on the relative position and proper velocity of the charge at the retarded τ. In is independent of the acceleration and of other properties of the trajectory. The retarded time τ is determined by the light-cone condition $R\bar{R} = 0$, where causality demands that $R^0 > 0$. To calculate A we can easily expand

$$\langle R\bar{u}\rangle_S = \gamma R^0 \left\langle \left(1 + \hat{\mathbf{R}}\right)(1 - \mathbf{v})\right\rangle_S = \gamma R^0 \left(1 - \hat{\mathbf{R}}\cdot\mathbf{v}\right), \tag{A4}$$

but this is not particularly helpful in elucidating the physics.

2. Liénard-Wiechert Field

The electromagnetic field is

$$\mathbf{F} = \left\langle\partial\bar{A}\right\rangle_V \equiv \partial\bar{A} - \left\langle\partial\bar{A}\right\rangle_S\ , \tag{A5}$$

with the added complication that the light-cone condition makes τ a function of x, that is a scalar field, so that there are two contributions to the gradient term $\partial \equiv \mathbf{e}^{\mu}\partial/\partial x^{\mu}$ (summation over $\mu = 0, 1, 2, 3$, is assumed)

$$\partial = (\partial)_{\tau} + [\partial\tau(x)]\frac{d}{d\tau}\ . \tag{A6}$$

The first term is differentiation with respect to the field position x with τ held fixed. The second term arises from the dependence on x of the scalar field $\tau(x)$. If we apply this to the light-cone condition $R\bar{R} = 0$, we find directly

$$\partial\tau(x) = \frac{R}{\langle R\bar{u}\rangle_S}. \tag{A7}$$

The evaluation of the field is now straightforward:

$$\mathbf{F}(x) = \frac{Ke}{\langle R\bar{u}\rangle_S^3}\left(\langle R\bar{u}\rangle_V + \frac{1}{2}R\bar{\dot{u}}u\bar{R}\right) \tag{A8}$$

$$= \mathbf{F}_c + \mathbf{F}_r \tag{A9}$$

with $\langle R\bar{u}\rangle_V = \left(R\bar{u} - u\bar{R}\right)/2$ the vector part of $R\bar{u}$. The parts have simple interpretations:

$$\mathbf{F}_c = \frac{Ke}{\langle R\bar{u}\rangle_S^3}\langle R\bar{u}\rangle_V \tag{A10}$$

is the boosted Coulomb field that falls off as $(R^0)^{-2}$ and

$$\mathbf{F}_r = \frac{Ke}{\langle R\bar{u}\rangle_S^3}\frac{R\bar{\dot{u}}u\bar{R}}{2} = \left(1+\hat{\mathbf{R}}\right)\frac{Ke\,(R^0)^2}{\langle R\bar{u}\rangle_S^3}\left(\bar{\dot{u}}u\right)_\perp \tag{A11}$$

is a directed wave propagating in the direction $\hat{\mathbf{R}}$ that falls off as $(R^0)^{-1}$ and is linear in the acceleration. Because $\bar{u}u = 1$, the factor $\bar{\dot{u}}u = -\bar{u}\dot{u} = \left\langle\bar{\dot{u}}u\right\rangle_V$ is vectorlike, and $\left(\bar{\dot{u}}u\right)_\perp$ is its part orthogonal to $\hat{\mathbf{R}}$, since any part that commutes with $\mathbf{R}$ gets killed by $R\bar{R} = 0$. The ratio $\langle R\bar{u}\rangle_V / \langle R\bar{u}\rangle_S$ is a unit vector, equal to $\hat{\mathbf{R}}$ when $u = 1$ but generally complex, indicating that it has a magnetic component. Both $\mathbf{F}_c$ and $\mathbf{F}_r$ are simple fields (single spacetime planes), with $\mathbf{F}_c$ predominantly electric, $\mathbf{F}_c \cdot \mathbf{F}_r = 0$, and $\mathbf{F}_r$ a null flag. The spacetime plane of $\mathbf{F}_c$ is the plane containing R and u. It is the R of the Coulomb field at rest, swept along u. Explicitly

$$\langle R\bar{u}\rangle_V = \gamma\left(\mathbf{R} - \mathbf{v}R^0 - \langle\mathbf{R}\mathbf{v}\rangle_V\right) \tag{A12}$$

The real part gives the electric field in the direction of $\mathbf{R}$ from the retarded position of the charge, minus $\mathbf{v}$ times the time R^0 for the radiation to get from the charge to the field point. The magnetic field lines are normal to the spatial plane $\langle\mathbf{R}\mathbf{v}\rangle_V$ swept out by $\mathbf{R}$ as the charge moves along $\mathbf{v}$. They thus circle around the charge path, as seen in Figure 2. With APS, we easily decipher both the spacetime geometry and the spatial version with the same formalism.

3. Virtual-Photon Method

Consider first the simple case that the charge is moving at constant velocity on its world line $r(\tau)$. The virtual ray which in the rest frame is directed along the lightlike paravector R_{rest} will generate a sheet

$$r(\tau) + \alpha R = r(\tau) + \alpha\Lambda R_{\text{rest}}\Lambda^\dagger, \tag{A13}$$

where $\alpha \geq 0$ and τ are the real scalar parameters of the sheet. Since both the spacetime direction R and the proper velocity u are constant, the sheet is coincident with the spacetime plane $\langle R\bar{u}\rangle_V$ and hence with the electromagnetic field $\mathbf{F} = \mathbf{F}_c$. In any inertial frame, a spacelike intersection of the sheets (at a given time in that frame) gives the electric-field lines of the charge.

More generally, the sheets swept out by the rays of an accelerating charge are not flat but curved, and the plane tangent to the sheet at $x = r + R$ on the ray along R emitted from the charge at $r(\tau)$ is given by the spacetime plane

$$\left\langle R\frac{d}{d\tau}\left(\bar{r} + \bar{R}\right)\right\rangle_V = \left\langle R\left(\bar{u} + \frac{d}{d\tau}\bar{R}\right)\right\rangle_V . \tag{A14}$$

In addition to the change in position r of the base of the ray, the spatial direction of the lightlike ray R may also change. Since the direction of the ray is fixed in the rest frame,

$$\frac{d}{d\tau}\bar{R} = \frac{d}{d\tau}\overline{\Lambda R_{\text{rest}}\Lambda^\dagger} = \overline{\langle \Omega R\rangle_{\Re}} , \tag{A15}$$

where $\Omega = 2\dot{\Lambda}\bar{\Lambda}$ is the spacetime rotation rate of the charge and $\langle x\rangle_{\Re} \equiv \left(x + x^\dagger\right)/2$ means the real part of x. Since $R\bar{R} = 0$ and the scalar part of Ω vanishes, the plane tangent to the electromagnetic field at x is $\langle R\bar{u}\rangle_V - \frac{1}{2}R\Omega^\dagger\bar{R}$. Consider the case that the acceleration is spatially collinear with the velocity. Then Ω is real and commutes with u, so that

$$\dot{u} = \dot{\Lambda}\Lambda^\dagger + \Lambda\dot{\Lambda}^\dagger = \langle \Omega u\rangle_{\Re} = \Omega u . \tag{A16}$$

In this case, $\Omega = \dot{u}\bar{u}$ and the tangent plane becomes $\langle R\bar{u}\rangle_V + \frac{1}{2}R\bar{\dot{u}}u\bar{R}$, which is just the spacetime plane of the electromagnetic field (A8). The electric field lines lie in the virtual photon sheets. They are generally no longer in the propagation direction $\hat{\mathbf{R}}$, but are formed by the intersection of the sheet with a spatial hypersurface at a given instant in time. The field line connects virtual-photon rays emitted in a given rest-frame direction from different positions on the world line of the charge.