Geometric Algebra: A Unified Mathematical Language for Physics

A Workshop at the AAPT Winter Meeting in Austin, Texas, January 12, 2003, organized in cooperation with *David Hestenes, Arizona State University.* W. E. *Baylis, University of Windsor, Ontario, Canada baylis@uwindsor.ca*

A Workbook

1 Overview of Geometric Algebra (GA)

- An algebra of real vectors (and their products)
- Simple formulation based on axiom: $\mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v}$
- Rich rewards:
  - Explains geometric meanings of cross product
  - Generalizes cross product to $n > 3$ dimensions → relativity
  - Clears potential confusion of pseudovectors and pseudoscalars
  - Constructs unit imaginary i as geometric object
  - Extends complex analysis to more than two dimensions
  - Reduces rotations, Lorentz transformations to algebraic multiplication
  - Gives classical spinors ("rotors") and projectors → quantum theory
  - Maximally exploits geometric properties and symmetries
  - Allows computational geometry without matrices or tensors
  - Treats Newtonian mechanics, relativity, and quantum theory with single formalism and language.

2 Essentials of Clifford's Geometric Algebra

2.1 The product

Multiplication in geometric algebra (GA) is like that for square matrices: it is associative and distributive over addition but generally not commutative, although multiplication by scalars commutes. *Collinear vectors* are related by a scalar. Thus, if $\mathbf{v}$ is any vector and λ and scalar, $\lambda\mathbf{v}$ is collinear with $\mathbf{v}$. It follows that collinear vectors commute:

$$(\lambda\mathbf{v})\,\mathbf{v} = \lambda\mathbf{v}\mathbf{v} = \mathbf{v}\,(\lambda\mathbf{v})\,.$$

The *meaning* of the "geometric product" of vectors in a space with a scalar product is fixed by the *defining axiom*

$$\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v}, \tag{1}$$

which with $\mathbf{v} = \mathbf{u} + \mathbf{w}$ implies

$$\mathbf{u}\mathbf{w} + \mathbf{w}\mathbf{u} = 2\mathbf{u}\cdot\mathbf{w}\ . \tag{2}$$

Exercise 1 *Prove (2) from the axiom (1) with* $\mathbf{v} = \mathbf{u} + \mathbf{w}$.

The *wedge product* (antisymmetric part of the product)

$$\mathbf{u}\wedge\mathbf{w} \equiv \mathbf{u}\mathbf{w} - \mathbf{u}\cdot\mathbf{w} = \frac{\mathbf{u}\mathbf{w} - \mathbf{w}\mathbf{u}}{2}$$

vanishes if and only if the vectors $\mathbf{u},\mathbf{w}$ are aligned.

Exercise 2 *Show that the bivector* $\mathbf{u}\wedge\mathbf{w}$ *anticommutes not only with* $\mathbf{u}$ *but also with* $\mathbf{w}$*, and therefore with any linear combination of* $\mathbf{u}$ *and* $\mathbf{w}$*, that is with any vector in the* plane *of* $\mathbf{u}$ *and* $\mathbf{w}$.

Exercise 3 *Prove that* $\mathbf{u} \wedge \mathbf{v}$ *is linear in both of its factors. Show, for example, that for any scalar* α *and vectors* $\mathbf{u}, \mathbf{w}, \mathbf{w}_1,$ *and* $\mathbf{w}_2$,

$$\begin{aligned} \mathbf{u} \wedge (\alpha \mathbf{w}) &= \alpha (\mathbf{u} \wedge \mathbf{w}) = (\alpha \mathbf{u}) \wedge \mathbf{w} \\ \mathbf{u} \wedge (\mathbf{w}_1 + \mathbf{w}_2) &= \mathbf{u} \wedge \mathbf{w}_1 + \mathbf{u} \wedge \mathbf{w}_2 \ . \end{aligned}$$

Any vector that is *orthogonal* to the plane, that is, orthogonal to both $\mathbf{u}$ and $\mathbf{w}$, anticommutes with both $\mathbf{u}$ and $\mathbf{w}$ and therefore *commutes with the product* $\mathbf{uw}$ and therefore also with the *bivector* $\mathbf{u} \wedge \mathbf{w}$.

Exercise 4 *Let* $\mathbf{v}$ *be orthogonal to both* $\mathbf{u}$ *and* $\mathbf{w}$*. Verify that* $\mathbf{v}$ *commutes with the bivector* $\mathbf{u} \wedge \mathbf{w}$.

2.2 Reversal and Hermitian Conjugation

An important conjugation (aka antiautomorphism, anti-involution) is reversal of the order of vector factors. In Euclidean spaces, it is convenient to denote the conjugation by a dagger:

$$(\mathbf{vw})^{\dagger} = \mathbf{w}^{\dagger}\mathbf{v}^{\dagger} = \mathbf{wv}.$$

Any element invariant under reversal is said to be *real* whereas elements that change sign are imaginary. Any element x can be split into real and imaginary parts:

$$x = \frac{x + x^{\dagger}}{2} + \frac{x - x^{\dagger}}{2} \equiv \langle x \rangle_{\Re} + \langle x \rangle_{\Im} \ .$$

Scalars and vectors (in a Euclidean space) are thus real, whereas bivectors are imaginary.

Exercise 5 *Show that the dot and wedge product of any vectors* $\mathbf{u}, \mathbf{v}$, *can be identified as the real and imaginary parts of the geometric product* $\mathbf{uv}$.

2.3 Bases

While in GA we usually work directly with vectors without concern for specific components, some properties are simpler to envision in an orthonormal basis $\{\mathbf{e}_j\}$:

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + \cdots \equiv v^j\mathbf{e}_j \ ,$$

where the summation convention is assumed for repeated indices. In Euclidean space the defining axiom becomes

$$\mathbf{e}_j \cdot \mathbf{e}_k = \frac{1}{2}\left(\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j\right) = \delta_{jk} \ .$$

Thus, for example, $\mathbf{e}_1^2 = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 = \mathbf{e}_2^2$ and $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$.

A *bivector basis* is formed from the products of orthonormal basis vectors, such as

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \ .$$

Its square is -1 and is therefore called a *unit bivector*. It represents the *plane* containing all vectors that are linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$.

Exercise 6 *Verify that* $\left(\mathbf{e}_1\mathbf{e}_2\right)^2 = -1$.

In an n-dimensional space, a suitable bivector basis is

$$\{\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \ldots, \mathbf{e}_1\mathbf{e}_n, \mathbf{e}_2\mathbf{e}_3, \ldots\} .$$

It contains exactly $n(n-1)/2 = \binom{n}{2}$ linearly independent elements of *grade two* (bivectors).

Exercise 7 *Show that while the dimensionality of bivector space is the same as that of the vector space when $n = 3$, the number of linearly independent bivectors when $n = 4$ is greater than n, and that when $n = 5$, the bivector space has twice as many dimensions as the vector space.*

Exercise 8 *Consider the product $\mathbf{uw}$ of two vectors in the $\mathbf{e}_1\mathbf{e}_2$ plane, namely $\mathbf{u} = u^1\mathbf{e}_1 + u^2\mathbf{e}_2$ and $\mathbf{w} = w^1\mathbf{e}_1 + w^2\mathbf{e}_2$. Show that $\mathbf{uw} = \mathbf{u}\cdot\mathbf{w} + \mathbf{u}\wedge\mathbf{w}$ with*

$$\begin{aligned} \mathbf{u}\cdot\mathbf{w} &= \left(u^1w^1 + u^2w^2\right) \\ \mathbf{u}\wedge\mathbf{w} &= \left(u^1w^2 - u^2w^1\right)\mathbf{e}_1\mathbf{e}_2 \ . \end{aligned}$$

In the last exercise we saw that the bivector $\mathbf{u}\wedge\mathbf{w}$ is

$$\mathbf{u}\wedge\mathbf{w} = \mathbf{e}_1\mathbf{e}_2 \det\begin{pmatrix} u^1 & w^1 \\ u^2 & w^2 \end{pmatrix}. \tag{3}$$

Exercise 9 *Show that $\mathbf{u}\wedge\mathbf{w}$ is invariant under orthogonal transformations such as rotations in the $\mathbf{e}_1\mathbf{e}_2$ plane:*

$$\begin{pmatrix} u^1 & w^1 \\ u^2 & w^2 \end{pmatrix} \to \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} u^1 & w^1 \\ u^2 & w^2 \end{pmatrix}.$$

Evidently, $\mathbf{u} \wedge \mathbf{w}$ represents the plane containing $\mathbf{u}$ and $\mathbf{w}$; its magnitude is the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{w}$, and its sign depends on the *direction of circulation* in the plane (see Fig. 1). As the next problem shows, it does not depend on the actual orientation of $\mathbf{u}$ and $\mathbf{w}$ in the plane.

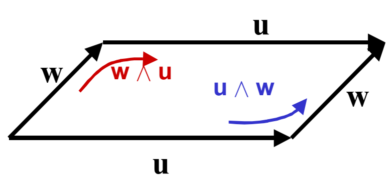

Figure 1: The wedge product of two vectors represents the plane containing the vectors. Its magnitude is the area of the parallelogram, and its sign indicates the circulation direction in the plane. The shape is not significant.

Exercise 10 *Define vectors* $\mathbf{s}$ *and* $\mathbf{u}$ *by*

$$\begin{aligned} \mathbf{s} &= \frac{\alpha}{2}(\mathbf{v}+\mathbf{w}) \\ \mathbf{u} &= (\mathbf{v}-\mathbf{w})/\alpha \end{aligned}$$

where α *is any scalar, and sketch the parallelograms formed by* $\mathbf{v}$ *and* $\mathbf{w}$ *and by* $\mathbf{s}$ *and* $\mathbf{u}$ *with* $\alpha = 1$. *Now show explicitly that* $\mathbf{s} \wedge \mathbf{u} = \mathbf{w} \wedge \mathbf{v}$. *[Hint: it is sufficient to note that the wedge product is antisymmetric and linear in its two factors.]*

The GA of n-D Euclidean space is denoted here by $\mathcal{C}\ell_n$ (the same as Hestenes' $\mathcal{G}_n$). More generally, $\mathcal{C}\ell_{p,q}$ denotes the GA of a pseudo-Euclidean space of signature (p,q) . Thus, $\mathcal{C}\ell_n \equiv \mathcal{C}\ell_{n,0}$.[1]

Exercise 11 *Consider the triangle of vectors* $\mathbf{c} = \mathbf{a} + \mathbf{b}$. *Prove*

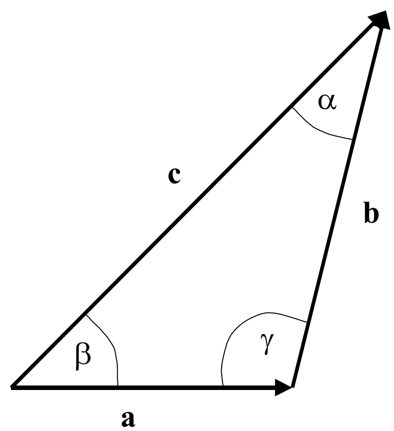

Figure 2: A triangle of vectors $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{c} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{c}$$

and show that the magnitude of these wedge products is twice the area of the triangle.

[1] The length of a vector $\mathbf{v} = v^k \mathbf{e}_k$ in a pseudo-Euclidean space can be expressed in the form

$$\mathbf{v} \cdot \mathbf{v} = \sum_{k=1}^{p} \left(v^k\right)^2 - \sum_{j=p+1}^{p+q} \left(v^j\right)^2 ,$$

and the space is then said to have signature (p,q) .

Exercise 12 *Let α, β, γ be the interior angles of the triangle (see last exercise) opposite sides* $\mathbf{a}, \mathbf{b}, \mathbf{c}$, *respectively. Use the relation of the wedge products to prove the law of sines:*

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c},$$

where a, b, c are the magnitudes of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Exercise 13 *Let* $\mathbf{r}$ *be the position vector of a point that moves with velocity* $\mathbf{v} = \dot{\mathbf{r}}$. *Show that the magnitude of the bivector* $\mathbf{r} \wedge \mathbf{v}$ *is twice the rate at which the time-dependent* $\mathbf{r}$ *sweeps out area. This relates the conservation of angular momentum* $\mathbf{r} \wedge \mathbf{p}$, *with* $\mathbf{p} = m\mathbf{v}$, *to Kepler's second law for planetary orbits, namely that equal areas are swept out in equal times.*

2.4 Existence and matrix representations

It's possible to define mathematical structures that are internally inconsistent and therefore don't exist. The easiest way to demonstrate self-consistency is to find a faithful *matrix representation* for the algebra.

The standard matrix representation of the algebra of physical (3-D Euclidean) space (APS) associates orthonormal basis vectors with Pauli spin matrices

$$\mathbf{e}_k \simeq \underline{\sigma}_k,\ k = 1, 2, 3.$$

Since the defining relations,

$$\frac{1}{2}\left(\underline{\sigma}_j\underline{\sigma}_k + \underline{\sigma}_k\underline{\sigma}_j\right) = \delta_{jk}\underline{1},$$

work, the algebra exists.

Exercise 14 *Using the standard Pauli-spin matrices*

$$\underline{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \underline{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \underline{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

verify the relation $\frac{1}{2}\left(\underline{\sigma}_j\underline{\sigma}_k + \underline{\sigma}_k\underline{\sigma}_j\right) = \delta_{jk}\underline{1}$ *for the cases* $j = k = 1$ *and* $j = 2$, $k = 3$.

Remark 1 *Physicists meeting GA for the first time often find it convenient to think of the algebra as an algebra of matrices. This was Pauli's and Dirac's approach to their algebras, but we advise against it. Thinking in terms of specific matrices is wasteful, often misleading, and unduly constraining: many matrix reps exist for each algebra; what is important are not the matrices but only their algebra.*

3 Bivectors as operators

The fact that the bivector of a plane commutes with vectors orthogonal to the plane and anticommutes with ones in the plane means that we can easily use unit bivectors to represent reflections. In particular, the two-sided transformation

$$\mathbf{v} \to \mathbf{e}_1\mathbf{e}_2\mathbf{v}\mathbf{e}_1\mathbf{e}_2$$

reflects any vector $\mathbf{v}$ in the $\mathbf{e}_1\mathbf{e}_2$ plane, as is verified in the next exercise.

Exercise 15 *Expand* $\mathbf{v} = v^k\mathbf{e}_k$ *in the* n*-dimensional basis* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ *to prove*

$$\mathbf{e}_1\mathbf{e}_2\mathbf{v}\mathbf{e}_1\mathbf{e}_2 = 2\left(v^1\mathbf{e}_1 + v^2\mathbf{e}_2\right) - \mathbf{v} = \mathbf{v}^{\triangle} - \mathbf{v}^{\perp},$$

where $\mathbf{v}^{\triangle}$ *is the component of* $\mathbf{v}$ *coplanar with* $\mathbf{e}_1\mathbf{e}_2$ *and* $\mathbf{v}^{\perp} = \mathbf{v} - \mathbf{v}^{\triangle}$ *is the component orthogonal to the plane. In words, components in the* $\mathbf{e}_1\mathbf{e}_2$ *plane*

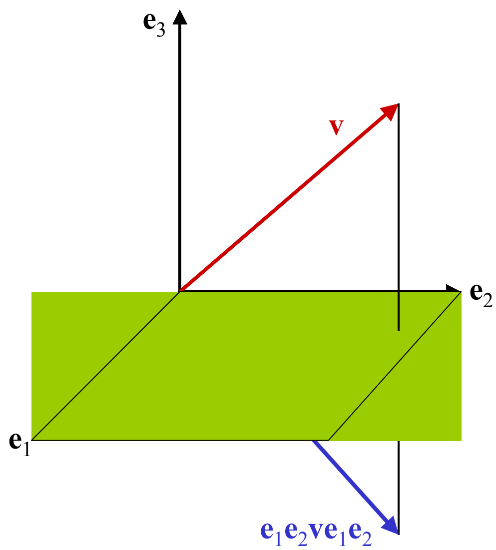

Figure 3: The reflection of $\mathbf{v}$ in the plane $\mathbf{e}_1\mathbf{e}_2$ is $\mathbf{e}_1\mathbf{e}_2\mathbf{v}\mathbf{e}_1\mathbf{e}_2$.

remain unchanged, but those orthogonal to the plane change sign. This is what we mean by a reflection in the $\mathbf{e}_1\mathbf{e}_2$ *plane.*

Exercise 16 *Show that the coplanar component of* $\mathbf{v}$ *is given by*

$$\mathbf{v}^{\triangle} = \frac{1}{2}\left(\mathbf{v} + \mathbf{e}_1\mathbf{e}_2\mathbf{v}\mathbf{e}_1\mathbf{e}_2\right).$$

Find a similar expression for the orthogonal component $\mathbf{v}^{\perp}$.

One of the most important properties of bivectors is that they generate rotations. To see this, try the following:

Exercise 17 *Simplify the products* $\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2)$ *and* $\mathbf{e}_2(\mathbf{e}_1\mathbf{e}_2)$.

Note that both $\mathbf{e}_1$ and $\mathbf{e}_2$ are rotated in the same direction through 90 degrees by right-multiplication with the bivector $\mathbf{e}_1\mathbf{e}_2$.It follows that the bivector is an operator on vectors in the plane: any vector $\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2$ in the $\mathbf{e}_1\mathbf{e}_2$ plane is rotated by 90° when multiplied by the unit bivector $\mathbf{e}_1\mathbf{e}_2$ (see Figure): $\mathbf{v} \rightarrow \mathbf{v}(\mathbf{e}_1\mathbf{e}_2)$.

Exercise 18 *Find the operator that upon multiplication from the right rotates any vector in the* $\mathbf{e}_1\mathbf{e}_2$ *plane by the small angle* $\alpha \ll 1$. *This should be expressed as a first-order approximation in* α .

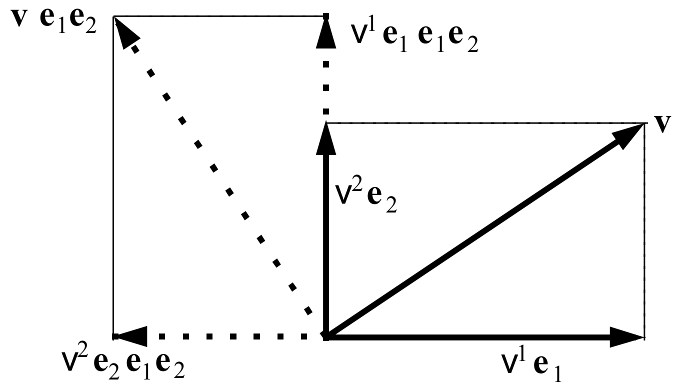

Figure 4: The bivector $\mathbf{e}_1\mathbf{e}_2$ rotates vectors in the $\mathbf{e}_1\mathbf{e}_2$ plane by 90°.

To rotate a vector by an angle θ other than 90°, use

$$\cos\theta + \mathbf{e}_1\mathbf{e}_2 \sin\theta = \exp(\mathbf{e}_1\mathbf{e}_2\theta) . \tag{4}$$

The *Euler relation* for the bivector follows from that for complex numbers: it depends only on $(\mathbf{e}_1\mathbf{e}_2)^2 = -1$. The bivector $\mathbf{e}_1\mathbf{e}_2$ thus *generates a rotation* in the $\mathbf{e}_1\mathbf{e}_2$ plane: for any vector $\mathbf{v}$ in the $\mathbf{e}_1\mathbf{e}_2$ plane, that vector is rotated by θ in the plane in the sense that takes $\mathbf{e}_1 \to \mathbf{e}_2$ by

$$\mathbf{v} \to \mathbf{v}\exp(\mathbf{e}_1\mathbf{e}_2\theta) = \exp(\mathbf{e}_2\mathbf{e}_1\theta)\,\mathbf{v} \ .$$

Exercise 19 *Verify that the operator* $\exp(\mathbf{e}_1\mathbf{e}_2\theta)$ *has the appropriate limits when* $\theta = 0$ *and when* $\theta = \pi/2$*, and that it also gives the correct linear approximation for small* θ*.*

The rotation is performed smoothly by increasing θ gradually from 0 to its full value. To represent a continual rotation in the $\mathbf{e}_1\mathbf{e}_2$ plane at the angular rate ω, we can let $\theta = \omega t$. Note that the rotation element $\exp(\mathbf{e}_1\mathbf{e}_2\theta)$ can also be expressed by

$$\exp(\mathbf{e}_1\mathbf{e}_2\theta) = \mathbf{e}_1\mathbf{e}_1 \exp(\mathbf{e}_1\mathbf{e}_2\theta) \equiv \mathbf{e}_1\mathbf{n} \ ,$$

where $\mathbf{n} = \mathbf{e}_1 \exp(\mathbf{e}_1\mathbf{e}_2\theta)$ is the unit vector obtained from $\mathbf{e}_1$ by a rotation of θ in the $\mathbf{e}_1\mathbf{e}_2$ plane.

Exercise 20 *Expand* $\mathbf{n} = \mathbf{e}_1 \exp(\mathbf{e}_1\mathbf{e}_2\theta)$ *in the basis* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *and verify that* $\mathbf{e}_1\mathbf{n} = \cos\theta + \mathbf{e}_1\mathbf{e}_2 \sin\theta$. *Show that the scalar and bivector parts of* $\exp(\mathbf{e}_1\mathbf{e}_2\theta)$ *are equal to* $\mathbf{e}_1 \cdot \mathbf{n}$ *and* $\mathbf{e}_1 \wedge \mathbf{n}$, *respectively.*

In general, every product $\mathbf{mn}$ of unit vectors $\mathbf{m}$ and $\mathbf{n}$ can be interpreted as a rotation operator of the form $\exp\left(\hat{\mathbf{B}}\theta\right)$, where the unit bivector $\hat{\mathbf{B}}$ represents the plane containing $\mathbf{m}$ and $\mathbf{n}$, and θ is the angle between them. The product $\mathbf{mn}$ therefore does not depend on the actual directions of $\mathbf{m}$ and $\mathbf{n}$, but only on the plane in which they lie and on the angle between them.

Exercise 21 *Let* $\mathbf{a} = \beta\mathbf{e}_1 \exp(\mathbf{e}_1\mathbf{e}_2\theta)$ *and* $\mathbf{b} = \beta^{-1}\mathbf{e}_2 \exp(\mathbf{e}_1\mathbf{e}_2\theta)$ *be vectors obtained by rotating* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *through the angle* θ *in the* $\mathbf{e}_1\mathbf{e}_2$ *plane and then dilating by complimentary factors. Prove that* $\mathbf{ab} = \mathbf{e}_1\mathbf{e}_2$ *.[Hint: note that* $\mathbf{b}$ *can also be written* $\beta^{-1} \exp(-\mathbf{e}_1\mathbf{e}_2\theta)\,\mathbf{e}_2$ *and that* $\exp(\mathbf{e}_1\mathbf{e}_2\theta)\exp(-\mathbf{e}_1\mathbf{e}_2\theta) = 1$.*]*

3.1 Relation to Complex Numbers

The complex number $r = x + iy$ corresponds directly not to $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$, but rather to $\mathbf{e}_1\mathbf{v} = x + y\mathbf{e}_1\mathbf{e}_2$:

$$\begin{aligned} i &\longleftrightarrow \mathbf{e}_1\mathbf{e}_2 \\ r &= x + iy \longleftrightarrow \mathbf{e}_1\mathbf{v} = x + y\mathbf{e}_1\mathbf{e}_2 \\ \bar{r} &= x - iy \longleftrightarrow \mathbf{v}\mathbf{e}_1 = x + y\mathbf{e}_2\mathbf{e}_1 \ . \end{aligned}$$

Exercise 22 *Use this correspondence to show*

$$\begin{aligned} r\bar{r}' &\longleftrightarrow \mathbf{e}_1\mathbf{v}\mathbf{v}'\mathbf{e}_1 = \mathbf{v}\cdot\mathbf{v}' - \mathbf{v}\wedge\mathbf{v}' \\ &= (xx' + yy') - (xy' - yx')\,\mathbf{e}_1\mathbf{e}_2 \ . \end{aligned}$$

Remark 2 *Geometric algebra explains why complex numbers work the way they do to represent vectors in two dimensional spaces. At the same time, it shows how they can be generalized to higher dimensions.*

3.2 Rotations in Spaces of More Than Two Dimensions

Exercise 23 *Use the Euler relation to expand the exponentials* $\exp(\mathbf{B}_1)$ *and* $\exp(\mathbf{B}_2)$ *of bivectors* $\mathbf{B}_1 = \theta_1\hat{\mathbf{B}}_1$ *and* $\mathbf{B}_2 = \theta_2\hat{\mathbf{B}}_2$*, where* $\hat{\mathbf{B}}_1^2 = \hat{\mathbf{B}}_2^2 = -1$*. Prove that if* $\hat{\mathbf{B}}_1 = \pm\hat{\mathbf{B}}_2$*, then*

$$\exp(\mathbf{B}_1)\exp(\mathbf{B}_2) = \exp(\mathbf{B}_1 + \mathbf{B}_2) \ .$$

Also show that when $\mathbf{B}_1\mathbf{B}_2 \neq \mathbf{B}_2\mathbf{B}_1$, *the relation* $\exp(\mathbf{B}_1)\exp(\mathbf{B}_2) = \exp(\mathbf{B}_1 + \mathbf{B}_2)$ *is not generally valid.*

In spaces of more than two dimensions, we can use

$$\mathbf{v} \rightarrow R\mathbf{v}R^{-1} \tag{5}$$

for the rotation, where $R = \exp(\mathbf{e}_2\mathbf{e}_1\theta/2)$ is a *rotor* and $R^{-1} = \exp(-\mathbf{e}_2\mathbf{e}_1\theta/2) = \exp(\mathbf{e}_1\mathbf{e}_2\theta/2)$ is its inverse.

Exercise 24 *Show that rotors in Euclidean spaces are unitary:* $R^{-1} = R^{\dagger}$.

Exercise 25 *Show that if* $\mathbf{n}$ *is the unit vector into which* $\mathbf{e}_1$ *is rotated by the rotor* $R = \exp(\mathbf{e}_2\mathbf{e}_1\theta/2)$, *that is* $\mathbf{n} = R\mathbf{e}_1R^{\dagger} = \exp(\mathbf{e}_2\mathbf{e}_1\theta)\,\mathbf{e}_1$, *then*

$$R = (\mathbf{n}\mathbf{e}_1)^{1/2} = \frac{(\mathbf{n}\mathbf{e}_1 + 1)}{\sqrt{2\,(1 + \mathbf{n}\cdot\mathbf{e}_1)}}\ .$$

[Hint: find the unit vector that bisects $\mathbf{n}$ *and* $\mathbf{e}_1$*.]*

3.3 Relation of Rotations to Reflections

Evidently $R^2 = \exp(\mathbf{e}_2\mathbf{e}_1\theta) = \mathbf{n}\mathbf{e}_1$ is the rotor for a rotation in the plane of $\mathbf{n}$ and $\mathbf{e}_1$ by 2θ. Its inverse is $\mathbf{e}_1\mathbf{n}$, and it takes any vector $\mathbf{v}$ into

$$\mathbf{v} \rightarrow \mathbf{n}\mathbf{e}_1\mathbf{v}\mathbf{e}_1\mathbf{n} \ .$$

If $\mathbf{e}_3$ is a unit vector normal to the plane of rotation, the rotation of $\mathbf{v}$ can be written as the result of reflections in two planes

$$\begin{aligned} \mathbf{v} &\rightarrow \mathbf{n}\mathbf{e}_3\mathbf{e}_3\mathbf{e}_1\mathbf{v}\mathbf{e}_3\mathbf{e}_1\mathbf{n}\mathbf{e}_3 \\ &= (\mathbf{n}\mathbf{e}_3)(\mathbf{e}_3\mathbf{e}_1)\,\mathbf{v}\,(\mathbf{e}_3\mathbf{e}_1)(\mathbf{n}\mathbf{e}_3) \end{aligned}$$

The unit bivectors of the two planes are $\mathbf{n}\mathbf{e}_3$ and $\mathbf{e}_3\mathbf{e}_1$. They intersect along $\mathbf{e}_3$ and have a dihedral angle of θ. See Fig. (5).

Example 1 *Mirrors in clothing stores are often arranged to give double reflections so that you can see yourself rotated rather than reflected. Two mirrors with a dihedral angle of* 90° *will rotate your image by* 180°*. This corresponds to the above transformation with* $\mathbf{n}$ *replaced by* $\mathbf{e}_2$*.*

Exercise 26 *How could you orient two mirrors so that you see yourself from the side, that is, rotated by* 270° *?*

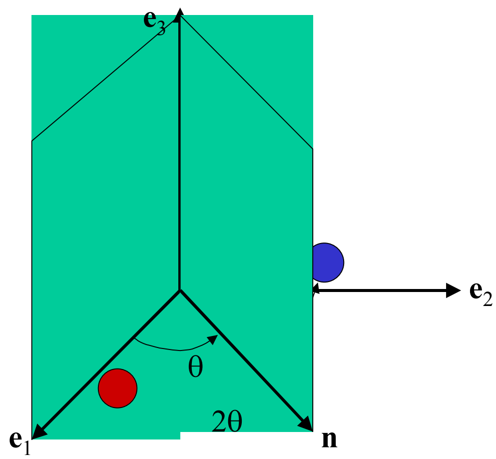

Figure 5: Successive reflections in two planes is equivalent to a rotation by twice the dihedral angle θ of the planes. In the diagram, the red ball is first reflected in the $\mathbf{e}_3\mathbf{e}_1$ plane and then in the $\mathbf{n}\mathbf{e}_3$ plane.

The rotation that results from successive reflections in two nonparallel planes in physical space depends only on the line of intersection and the dihedral angle between the planes; it is independent of rotations for both planes about their common axis.

Exercise 27 *Corner cubes are used on the moon and in the rear lenses on cars to reverse the direction of the incident light. Consider a sequence of three reflections, first in the* $\mathbf{e}_1\mathbf{e}_2$ *plane, followed by one in the* $\mathbf{e}_2\mathbf{e}_3$ *plane, followed by one in the* $\mathbf{e}_3\mathbf{e}_1$ *plane. Show that when applied to any vector* $\mathbf{v}$, *the result is* $-\mathbf{v}$.

The ability to rotate in any plane of n-dimensional space without components, tensors, or matrices is a major advantage of geometric algebra.

Exercise 28 *Show that the magnitude of* $\boldsymbol{\Theta}$ *in the rotor* $R = \exp(\boldsymbol{\Theta})$ *is the* area *swept out by any unit vector* $\mathbf{n}$ *in the rotation plane under the rotation* $\mathbf{n} \to R\mathbf{n}R^{-1}$. *[Hint: Note that the increment in area added when the unit vector is rotated through the incremental angle* $d\theta$ *is* $\frac{1}{2}d\theta$*.]*

Exercise 29 *Consider the Euler-angle rotor* $R = \exp\left(\mathbf{e}_2\mathbf{e}_1\frac{\phi}{2}\right)\exp\left(\mathbf{e}_1\mathbf{e}_3\frac{\theta}{2}\right)\exp\left(\mathbf{e}_2\mathbf{e}_1\frac{\psi}{2}\right)$. *Show that when* $\theta = 0$ *the result depends only on* $\phi + \psi$ *and is independent of the value* $\phi - \psi$, *whereas when* $\theta = \pi$, *the converse holds.*

3.4 Spatial Rotations as Spherical Vectors

Any rotation is specified by the plane of rotation and the area swept out by a unit vector in the plane under the rotation. As any rotation in physical (3-dimensional Euclidean) space proceeds, the unit vector sweeps out a path on the surface S^2 of a unit sphere, and this path serves to represent the rotation. Any plane containing one of the unit vectors includes the origin and intersects S^2 in a great circle. The path representing any rotation is a *directed arc* on such a great circle. We call such directed arcs *spherical vectors.* Spherical vectors are as straight as they can be on S^2, and they can be freely translated along their great circles. However, spherical vectors on different great circles represent rotations on different planes and are generally distinct.

Any rotor $R = \exp \mathbf{\Theta}$ is the product of two unit vectors $\mathbf{a}, \mathbf{b}$,

$$R = \exp \mathbf{\Theta} = \mathbf{ba} \ ,$$

where both $\mathbf{a}$ and $\mathbf{b}$ lie in the plane of $\mathbf{\Theta}$ and the angle between them is Θ. (Points on S^2 are of course the ends of unit vectors from the origin of the sphere; we generally represent a unit vector and its point of intersection on S^2 with the same bold-face symbol.) The spherical vector $\overrightarrow{\mathbf{ab}}$ on S^2 from $\mathbf{a}$ to $\mathbf{b}$ represents R.

Exercise 30 *If $R = \mathbf{ba}$, then $R^\dagger = \mathbf{ab}$. Verify with these expressions that R and $R^\dagger$ are inverses of each other.*

Exercise 31 *Show that $\mathbf{ba} = \left(R\mathbf{b}R^{-1}\right)\left(R\mathbf{a}R^{-1}\right)$ for any rotor $R = \exp\left(\alpha\hat{\mathbf{\Theta}}\right)$ in the plane of $\mathbf{a}$ and $\mathbf{b}$.*

Now let's combine R with a rotation in a different plane, say R'. Distinct planes have distinct great circles on S^2 and intersect at antipodal points. We slide $\mathbf{a}$ and $\mathbf{b}$ along the great circle of $\mathbf{\Theta}$ until $\mathbf{b}$ is at one of the intersections. Then we can choose $\mathbf{c}$ so that $R' = \mathbf{cb}$ and the composition

$$R'R = \mathbf{cb}\ \mathbf{ba} = \mathbf{ca}$$

yields a rotation represented by the spherical vector $\overrightarrow{\mathbf{ac}} = \overrightarrow{\mathbf{ab}} + \overrightarrow{\mathbf{bc}}$ from $\mathbf{a}$ to $\mathbf{c}$.The composition of rotations thus is equivalent to the *addition of spherical vectors* on S^2 (see Fig. 6).

The length of the spherical vector $\overrightarrow{\mathbf{ab}}$ from $\mathbf{a}$ to $\mathbf{b}$, which represents the rotor $R = \mathbf{ba}$, is *half* the maximum angle of rotation of a vector

$$\mathbf{v} \to R\mathbf{v}R^{-1}.$$

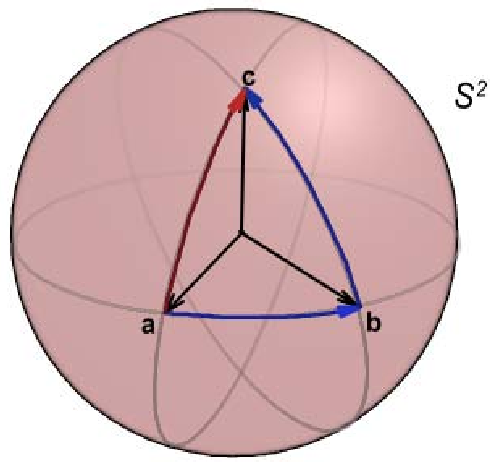

Figure 6: A product of rotations is represented by the addition of spherical vectors.

In other words, the *length* of $\overrightarrow{\mathbf{ab}}$ is the *area* swept out by a *unit* vector in the rotation plane. Points on S^2 are not directly associated with directions in physical space. *Pairs* of points on S^2 separated by an angle θ represent rotors in physical space that rotate vectors by up to an angle of 2θ, whereas the points themselves are associated with *spinors.* (The points do not fully identify the spinors but only their *poles.* Their orientation about the poles requires an additional complex phase which is not required for the treatment of rotations.)

We refer to S^2 as the *Cartan sphere.*[2] It is not to be confused with the unit sphere in physical space. Indeed, there is a two-to-one mapping of points from S^2 to directions in physical space. Antipodes on the Cartan sphere map to the same direction in physical space. [Spherical vectors on S^2 give a faithful representation of rotations in $SU(2)$, the double covering group of $SO(3)$.] Note that the addition of spherical vectors is noncommutative. This reflects the noncommutivity of rotations in different planes. See for example Fig.7.

Example 2 *What's the product of a* 180° *rotation in the* $\mathbf{e}_2\mathbf{e}_1$ *plane followed by a* 180° *rotation in the* $\mathbf{e}_3\mathbf{e}_2$ *plane? Use the Euler relation* $\exp(\mathbf{e}_2\mathbf{e}_1\alpha) = \cos\alpha + \mathbf{e}_2\mathbf{e}_1 \sin\alpha$ *to get*

$$\exp\left(\mathbf{e}_3\mathbf{e}_2\frac{\pi}{2}\right)\exp\left(\mathbf{e}_2\mathbf{e}_1\frac{\pi}{2}\right) = \mathbf{e}_3\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3\mathbf{e}_1 = \exp\left(\mathbf{e}_3\mathbf{e}_1\frac{\pi}{2}\right).$$

The result is therefore a 180° *rotation in the* $\mathbf{e}_3\mathbf{e}_1$ *plane. Note that we do not need to compute an entire rotation* $R\mathbf{v}R^{-1}$ *but only the rotor* R*. In terms of*

[2] In recognition of Élie Cartan's extensive work with spinor representations of simple Lie algebras.

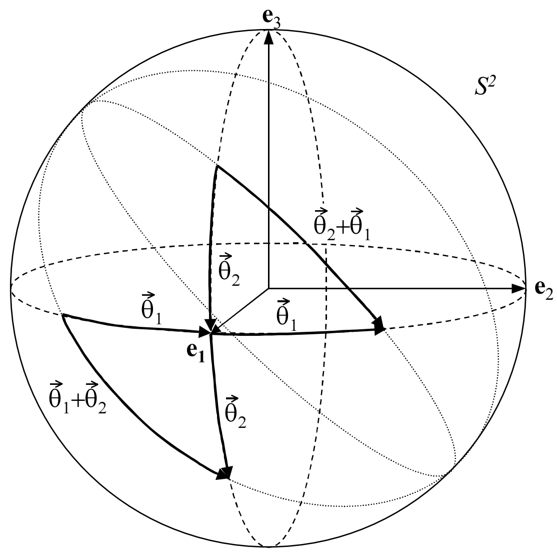

Figure 7: Addition of spherical vectors is not commutative. The sum $\vec{\theta}_1 + \vec{\theta}_2$ is on a different great circle than $\vec{\theta}_2 + \vec{\theta}_1$.

spherical vectors, the composition is equivalent to adding a 90° *vector on the equator to one joining the equator to the north pole.*

Exercise 32 *Show that the result of a* 90° *rotation in the* $\mathbf{e}_1\mathbf{e}_2$ *plane followed by a* 90° *rotation in the* $\mathbf{e}_2\mathbf{e}_3$ *plane is a* 120° *rotation in the plane*

$$(\mathbf{e}_1\mathbf{e}_2+\mathbf{e}_2\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_1)/\sqrt{3} = \frac{1}{2\sqrt{3}}(\mathbf{e}_1-\mathbf{e}_2)(\mathbf{e}_1+\mathbf{e}_2-2\mathbf{e}_3).$$

We now have both an algebraic way and a geometric way to rotate any vector by any angle in any plane, and the relation provides simple calculations in geometric algebra for manipulations in spherical trigonometry. If $\hat{\mathbf{B}}$ is the unit bivector for the plane and θ is the angle of rotation, the vector $\mathbf{v}$ under rotation becomes

$$\begin{aligned} \mathbf{v} &\rightarrow \mathbf{v}' = R\mathbf{v}R^{-1} \\ R &= \exp\left(\hat{\mathbf{B}}\theta/2\right). \end{aligned}$$

If, for example, $\hat{\mathbf{B}} = \mathbf{e}_2\mathbf{e}_1$, the sense of the rotation is from $\mathbf{e}_1$ towards $\mathbf{e}_2$. The rotation can be evaluated algebraically without the need for components or matrices. While one can expand $R = \cos\theta/2 + \hat{\mathbf{B}}\sin\theta/2$, it is much easier to first expand $\mathbf{v}$ into components in the plane of rotation (coplanar: $\triangle$) and orthogonal ($\perp$) to it:

$$\mathbf{v} = \mathbf{v}^{\triangle} + \mathbf{v}^{\perp}.$$

Since $\mathbf{v}^{\triangle}$ anticommutes with $\hat{\mathbf{B}}$ whereas $\mathbf{v}^{\perp}$ commutes with it,

$$\begin{aligned} R\mathbf{v}R^{-1} &= R^2\mathbf{v}^{\triangle} + \mathbf{v}^{\perp} \\ &= \mathbf{v}^{\triangle}\cos\theta + \hat{\mathbf{B}}\mathbf{v}^{\triangle}\sin\theta + \mathbf{v}^{\perp}. \end{aligned}$$

As before, a unit bivector times a vector in the plane of the bivector rotates that vector by a right angle in the plane.

Exercise 33 *Expand R^{-1} to prove that $\mathbf{v}^{\triangle}R^{-1} = R\mathbf{v}^{\triangle}$ and $\mathbf{v}^{\perp}R^{-1} = R^{-1}\mathbf{v}^{\perp}$, where $\mathbf{v}^{\triangle}$ lies in the plane of the rotation (is coplanar) and $\mathbf{v}^{\perp}$ is orthogonal to the rotation plane.*

Remark 3 *Unit bivectors in physical space ($n = 3$) can be identified with quaternion units: $\mathbf{i} = \mathbf{e}_3\mathbf{e}_2, \mathbf{j} = \mathbf{e}_1\mathbf{e}_3, \mathbf{k} = \mathbf{e}_2\mathbf{e}_1$. Rotors R are then unit quaternions. Rotations in physical space can thus be represented in the quaternion algebra. The geometrical interpretation is somewhat skewed since vectors in the algebra of physical space (APS) are not part of the quaternion algebra, but must instead be represented by their dual planes. Quaternions are popular in the aerospace and computer-games industries for representing rotations.*

Exercise 34 *With the identification $\mathbf{i} = \mathbf{e}_3\mathbf{e}_2, \mathbf{j} = \mathbf{e}_1\mathbf{e}_3, \mathbf{k} = \mathbf{e}_2\mathbf{e}_1$, show explicitly that*

$$\mathbf{ijk} = \mathbf{kk} = -1 .$$

3.5 Time-dependent Rotations

An additional infinitesimal rotation by $\mathbf{\Omega}dt$ during the time interval dt changes a rotor R to

$$R + dR = \left(1 + \frac{1}{2}\mathbf{\Omega}dt\right) R \ .$$

The time-rate of change of R thus has the form

$$\dot{R} = \frac{dR}{dt} = \frac{1}{2}\mathbf{\Omega}R,$$

where the bivector $\mathbf{\Omega}$ is the *rotation rate*. For the special case of a constant rotation rate, we can take the rotor to be

$$R(t) = e^{\mathbf{\Omega}t/2}.$$

Any vector $\mathbf{r}$ is thereby rotated to

$$\mathbf{r}' = R\mathbf{r}R^{-1}$$

giving a time derivative

$$\begin{aligned} \dot{\mathbf{r}}' &= R\left[\frac{\mathbf{\Omega r} - \mathbf{r\Omega}}{2} + \dot{\mathbf{r}}\right]R^{-1} \\ &= R\left[\langle\mathbf{\Omega r}\rangle_{\Re} + \dot{\mathbf{r}}\right]R^{-1}, \end{aligned}$$

where we noted that $\mathbf{r}$ is real and the bivector $\mathbf{\Omega}$ is imaginary. Since $\mathbf{r}$ can be any vector, we can replace it by $\langle\mathbf{\Omega r}\rangle_{\Re} + \dot{\mathbf{r}}$ to determine the second derivative

$$\begin{aligned} \ddot{\mathbf{r}}' &= R\left[\langle\mathbf{\Omega}\left(\langle\mathbf{\Omega r}\rangle_{\Re} + \dot{\mathbf{r}}\right)\rangle_{\Re} + \langle\mathbf{\Omega}\dot{\mathbf{r}}\rangle_{\Re} + \ddot{\mathbf{r}}\right]R^{-1} \\ &= R\left[\langle\mathbf{\Omega}\langle\mathbf{\Omega r}\rangle_{\Re}\rangle_{\Re} + 2\langle\mathbf{\Omega}\dot{\mathbf{r}}\rangle_{\Re} + \ddot{\mathbf{r}}\right]R^{-1}. \end{aligned} \tag{6}$$

Let's write $\mathbf{\Omega} = \omega\hat{\mathbf{\Omega}}$. Then $\langle\mathbf{\Omega r}\rangle_{\Re} = \mathbf{\Omega r}^{\triangle} = \omega\hat{\mathbf{\Omega}}\mathbf{r}^{\triangle}$, where $\mathbf{r}^{\triangle}$ is the part of $\mathbf{r}$ coplanar with $\hat{\mathbf{\Omega}}$. The product $\hat{\mathbf{\Omega}}\mathbf{r}^{\triangle}$ is $\mathbf{r}^{\triangle}$ rotated by a right angle in the plane $\hat{\mathbf{\Omega}}$.

Exercise 35 *Show that* $\langle \boldsymbol{\Omega} \langle \boldsymbol{\Omega} \mathbf{r} \rangle_{\Re} \rangle_{\Re} = -\omega^2 \mathbf{r}^{\triangle}$ *and that the minus sign can be viewed as arising from two 90-degree rotations or, equivalently, from the square of a unit bivector.*

The result can be expressed

$$\ddot{\mathbf{r}}' = R \left[-\omega^2 \mathbf{r}^{\triangle} + 2\omega \hat{\boldsymbol{\Omega}} \dot{\mathbf{r}}^{\triangle} + \ddot{\mathbf{r}} \right] R^{-1}$$

which clearly shows the transformation from the position vector $\mathbf{r}$ in a rotating frame to $\mathbf{r}'$ in an inertial system. A force law $\mathbf{f}' = m\ddot{\mathbf{r}}'$ in the inertial system is seen to be equivalent to an effective force

$$\mathbf{f} = m\ddot{\mathbf{r}} = R^{-1} \mathbf{f}' R + m\omega^2 \mathbf{r}^{\triangle} + 2m\omega \dot{\mathbf{r}}^{\triangle} \hat{\boldsymbol{\Omega}}$$

in the rotating frame. The second and third terms on the RHS are identified as the centrifugal and Coriolis forces, respectively.

4 Trivectors, Higher-Grade Multivectors, and Duals

Higher-order products vectors are readily formed. Products of k orthonormal basis vectors $\mathbf{e}_j$ can be reduced if two of them are the same. If they are all distinct, their product is a basis k-vector. Trivectors can be expanded in a basis $\{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_4, \cdots\}$ comprising products of three *distinct* basis vectors. In an n-dimensional vector space, there are $\binom{n}{3} = n(n-1)(n-2)/3!$ of these. The algebra contains 1 linearly independent scalar, n linearly independent vectors, $\binom{n}{2}$ linearly independent bivectors, $\binom{n}{3}$ linearly independent trivectors, and more generally $\binom{n}{k}$ linearly independent multivectors of *grade* k,for a total of

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

linearly independent elements. The highest grade element is the volume element, proportional to

$$\mathbf{e}_T \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \cdots \mathbf{e}_n \ .$$

Exercise 36 *Find the number of independent elements in the geometric algebra of $C\ell_5$ of 5-dimensional space. How is this subdivided into vectors, bivectors, and so on?*

The general element of the algebra contains a mixture of different grades. It is often useful to isolate parts of different grades. For this we use the notation $\langle x \rangle_k$ to indicate that part of x that has grade k. Thus, $\langle x \rangle_0$ is the scalar part of x, $\langle x \rangle_1$ is the vector part, and $\langle x \rangle_{2,1}$ is the sum of the bivector and vector parts. Evidently

$$x = \sum_{k=0}^{n} \langle x \rangle_k \ .$$

The exterior product of k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, is the k-grade part of the product:

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_k \equiv \langle \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_k \rangle_k \ .$$

It represents the k-volume contained in the polygon with parallel edges given by the vector factors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, and it vanishes unless all k vectors are linearly independent. In APS, in addition to scalars, vectors, and bivectors, there are also trivectors, elements of grade 3:

$$\begin{aligned} \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} &\equiv \langle \mathbf{uvw} \rangle_3 \\ &= \sum_{jkl} u^j v^k w^l \langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3 \\ &= \sum_{jkl} \varepsilon_{jkl} u^j v^k w^l \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \\ &= \mathbf{e}_T \det \begin{pmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{pmatrix}, \end{aligned} \tag{7}$$

where we noted that in 3-dimensional space, the 3-vectors $\langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3$ are related to the volume element $\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_T$ by the Levi-Civita symbol ε_{jkl} :

$$\langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3 = \varepsilon_{jkl} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \ . \tag{8}$$

Exercise 37 *Verify the relation (8) for several values of j, k, l.*

Note the appearance of the determinant in expression (7), as in a previous component form (3) of the bivector. It ensures that the wedge product vanishes if the vector factors are linearly dependent.

While the component expressions can be useful for comparing results with other work, the component-free versions $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} \equiv \langle \mathbf{uvw} \rangle_3$ are simpler and more efficient to work with. The factor $\mathbf{uv}$ can be split into scalar (grade-0) and bivector (grade-2) parts

$$\mathbf{uv} = \langle \mathbf{uv} \rangle_0 + \langle \mathbf{uv} \rangle_2 \,,$$

but $\langle \mathbf{uv} \rangle_0 \mathbf{w}$ is a (grade-1) vector, so that only the bivector piece contributes to the trivector $\langle \mathbf{uvw} \rangle_3$. Thus,

$$\langle \mathbf{uvw} \rangle_3 = \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_3 \,.$$

Now split $\mathbf{w}$ into components coplanar with $\langle \mathbf{uv} \rangle_2$ and orthogonal to it:

$$\mathbf{w} = \mathbf{w}^{\triangle} + \mathbf{w}^{\perp},$$

where $\mathbf{w}^{\triangle}$ and $\mathbf{w}^{\perp}$ are the parts that anticommute and commute with $\langle \mathbf{uv} \rangle_2$, respectively. The coplanar part $\mathbf{w}^{\triangle}$ is linearly dependent on $\mathbf{u}$ and $\mathbf{v}$ and therefore does not contribute to the trivector, whereas the part $\mathbf{w}^{\perp}$ orthogonal to $\mathbf{u}$ and $\mathbf{v}$ does. We are left with

$$\begin{aligned} \langle \mathbf{uvw} \rangle_3 &= \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_3 = \frac{1}{2} \left(\langle \mathbf{uv} \rangle_2 \mathbf{w} + \mathbf{w} \langle \mathbf{uv} \rangle_2 \right) \\ &= \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\Im} \;. \end{aligned}$$

It follows that the vector part of the product $\langle \mathbf{uv} \rangle_2 \mathbf{w}$ is

$$\begin{aligned} \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_1 &= \langle \mathbf{uv} \rangle_2 \mathbf{w} - \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_3 = \frac{1}{2} \left(\langle \mathbf{uv} \rangle_2 \mathbf{w} - \mathbf{w} \langle \mathbf{uv} \rangle_2 \right) \\ &= \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\Re} \;. \end{aligned}$$

A couple of important results follow easily.

Theorem 1 $\langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\Re} = \mathbf{u} \left(\mathbf{v} \cdot \mathbf{w} \right) - \mathbf{v} \left(\mathbf{u} \cdot \mathbf{w} \right).$

Proof. Expand $\langle\langle\mathbf{uv}\rangle_2\,\mathbf{w}\rangle_{\Re} = \frac{1}{2}\left(\langle\mathbf{uv}\rangle_2\,\mathbf{w} - \mathbf{w}\,\langle\mathbf{uv}\rangle_2\right)$, add and subtract term $\mathbf{uwv}$ and $\mathbf{vwu}$, and collect:

$$\begin{aligned}
\langle\langle\mathbf{uv}\rangle_2\,\mathbf{w}\rangle_{\Re} &= \frac{1}{4}\left[(\mathbf{uv}-\mathbf{vu})\,\mathbf{w} - \mathbf{w}\,(\mathbf{uv}-\mathbf{vu})\right] \\
&= \frac{1}{4}\left[\mathbf{uvw}+\mathbf{uwv}-\mathbf{vuw}-\mathbf{vwu}-\mathbf{wuv}-\mathbf{uwv}+\mathbf{wvu}+\mathbf{vwu}\right] \\
&= \frac{1}{2}\left[\mathbf{u}\,(\mathbf{v}\cdot\mathbf{w}) - \mathbf{v}\,(\mathbf{u}\cdot\mathbf{w}) - (\mathbf{w}\cdot\mathbf{u})\,\mathbf{v} + (\mathbf{w}\cdot\mathbf{v})\,\mathbf{u}\right] \\
&= \mathbf{u}\,(\mathbf{v}\cdot\mathbf{w}) - \mathbf{v}\,(\mathbf{u}\cdot\mathbf{w})\,.
\end{aligned}$$

■

Note that the vector $\langle\langle\mathbf{uv}\rangle_2\,\mathbf{w}\rangle_1$ lies in the plane of $\langle\mathbf{uv}\rangle_2$ and is orthogonal to $\mathbf{w}$. It corresponds to the *contraction* of a bivector with a vector and is sometimes written

$$\langle\langle\mathbf{uv}\rangle_2\,\mathbf{w}\rangle_1 = (\mathbf{u}\wedge\mathbf{v})\cdot\mathbf{w},$$

where the parenthesis is often dropped with the understanding that wedge products are evaluated before dot product. It lies in the *intersection* of the plane of $\langle\mathbf{uv}\rangle_2$ with the hypersurface dual to $\mathbf{w}$.

Since a vector $\mathbf{v}$ orthogonal to the space spanned by the vectors comprising a k-vector $\mathbf{K}$ commutes with $\mathbf{K}$ if k is even and anticommutes with it if k is odd, we can generalize the result for the trivector to

Theorem 2 *The $(k+1)$-vector $\langle\mathbf{Kv}\rangle_{k+1}$ is given in terms of the k-vector $\mathbf{K}$ as*

$$\langle\mathbf{Kv}\rangle_{k+1} = \frac{1}{2}\left(\mathbf{Kv} + (-)^k\,\mathbf{vK}\right).$$

Corollary 3 $\langle\mathbf{Kv}\rangle_{k-1} = \frac{1}{2}\left(\mathbf{Kv} - (-)^k\,\mathbf{vK}\right).$

4.1 Duals

In the algebra of an n-dimensional space, the number of independent k-grade multivectors is the same as the number of $(n-k)$-grade elements. Thus, both the vectors (grade 1 elements) and the pseudovectors (grade $n-1$ elements) occupy linear spaces of n dimensions. We can therefore establish a one-to-one mapping between such elements. We define the Clifford dual $^*x$ of an element x by

$$^*x \equiv x\mathbf{e}_T^{-1}.$$

The dual of a dual is $\mathbf{e}_T^{-2} = \pm 1$ times the original element. If x is a k-vector, each term in a k-vector basis expansion of x will cancel k of the basis vector factors in $\mathbf{e}_T$, leaving $^*x = x\mathbf{e}_T^{-1}$ as an $(n-k)$-vector. Furthermore, any simple element and its dual are fully orthogonal in the following sense: a simple k-vector is a single product of k independent vectors that span a k-dimensional subspace; every vector in that subspace is orthogonal to vectors whose products comprise the $(n-k)$-vector $^*x$. The dual of a scalar is a volume element, known as a *pseudoscalar*; the dual of a vector is the hypersurface orthogonal to that vector, known as a *pseudovector*; and so on.

In APS, the dual to a bivector is the vector normal to the plane of the bivector. Thus, $\mathbf{e}_T = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ and $\mathbf{e}_T^{-1} = \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_T$ and for example

$$^*(\mathbf{e}_1\mathbf{e}_2) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3 \ .$$

We recognize that the dual of a bivector in APS is the *cross product*:

$$^*(\mathbf{u} \wedge \mathbf{v}) = \mathbf{u} \times \mathbf{v} \ ,$$

and with this we can understand the relation between the cross product $\mathbf{u} \times \mathbf{v}$ and the plane of $\mathbf{u}$ and $\mathbf{v}$. The volume element in physical space squares to -1 and commutes with all vectors and hence all elements. It can therefore be associated with the *unit imaginary*:

$$\mathbf{e}_T = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = i$$

and thus $^*(\mathbf{u} \wedge \mathbf{v}) = (\mathbf{u} \wedge \mathbf{v})/i$ so that

$$\mathbf{u} \wedge \mathbf{v} = i\mathbf{u} \times \mathbf{v} \ .$$

However, whereas the cross product is defined only in three dimensions and is nonassociative as well as noncommutative, the exterior wedge product is defined in spaces of any dimension and is associative. It also emphasizes the essential properties of the plane and is an operator on vectors that generates rotations.

Remark 4 *APS thus automatically incorporates complex numbers as its* center *(commuting part). The unit imaginary has geometric meaning in the algebra: it is the unit volume element and enters in the dual relationship. This helps make sense of some of the many complex numbers that appear in* real *physics. The bivector, for example, is a pseudovector, the dual to the normal vector:*

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_3 = i\mathbf{e}_3 \ .$$

Exercise 38 *Show by calculation of some explicit values that the Levi-Civita symbol is the dual to the volume element* $\langle \mathbf{e}_j\mathbf{e}_k\mathbf{e}_l \rangle_3$ *in APS:*

$$\varepsilon_{jkl} = {}^*\langle \mathbf{e}_j\mathbf{e}_k\mathbf{e}_l \rangle_3 = \langle \mathbf{e}_j\mathbf{e}_k\mathbf{e}_l \rangle_3 \mathbf{e}_T^{-1}.$$

This definition is easily extended to higher dimensions.

We can use duals to express rotors in physical space in terms of the *axis of rotation.* For example

$$R = \exp(\mathbf{e}_2\mathbf{e}_1\theta/2) = \exp(-i\mathbf{e}_3\theta/2)$$

is the rotor for a rotation $\mathbf{v} \to R\mathbf{v}R^{-1}$ by θ about the $\mathbf{e}_3$ axis in physical space.

Exercise 39 *Express the bivector rotation rate* $\mathbf{\Omega} = -i\boldsymbol{\omega}$ *as the dual of a vector* $\boldsymbol{\omega}$ *in physical space. Show that* $\langle \mathbf{\Omega}\mathbf{r}\rangle_{\Re} = \boldsymbol{\omega} \times \mathbf{r}$.

Exercise 40 *Show that in physical space the theorem* $\langle \langle \mathbf{uv}\rangle_2 \mathbf{w}\rangle_{\Re} = \mathbf{u}(\mathbf{v}\cdot\mathbf{w}) - \mathbf{v}(\mathbf{u}\cdot\mathbf{w})$ *is equivalent to* $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \mathbf{u}\cdot\mathbf{w}\ \mathbf{v} - \mathbf{v}\cdot\mathbf{w}\ \mathbf{u}$.

Exercise 41 *Rewrite the transformation (6) from the rotating frame to the inertial lab frame in terms of* $\boldsymbol{\omega} = i\mathbf{\Omega}$.

The wedge product of a j-vector $\mathbf{J}$ with a k-vector $\mathbf{K}$ is the $(j+k)$-vector

$$\mathbf{J} \wedge \mathbf{K} = \langle \mathbf{JK}\rangle_{j+k}$$

Its dual is the $(n-j-k)$-vector

$$\langle \mathbf{JK}\rangle_{j+k}\,\mathbf{e}_T^{-1} = \left\langle \mathbf{JKe}_T^{-1}\right\rangle_{n-j-k} = \langle \mathbf{J}\ {}^*\mathbf{K}\rangle_{n-k-j}\ ,$$

where ${}^*\mathbf{K} = \mathbf{Ke}_T^{-1}$ is the dual of $\mathbf{K}$ and we have assumed $n \geq j+k$. The grades are specified here, but more generally we can write the wedge ("exterior") theorem

Theorem 4 *The dual of* $\mathbf{J}\wedge\mathbf{K}$ *is the contraction*

$${}^*(\mathbf{J}\wedge\mathbf{K}) = \mathbf{J}\cdot{}^*\mathbf{K}\ .$$

Corollary 5 *The dual of a contraction is the wedge with a dual, and if* $k \geq j$,

$${}^*(\mathbf{J}\cdot\mathbf{K}) = \mathbf{J}\wedge\ {}^*\mathbf{K}.$$

This provides a way of replacing contractions with wedge products (or vice versa).

Exercise 42 *The area of the parallelogram with sides* $\mathbf{v},\mathbf{w}$ *is the magnitude of the wedge product*

$$\mathbf{v}\wedge\mathbf{w} = \frac{1}{2}(\mathbf{vw}-\mathbf{wv})\,.$$

Insert $1 = -\mathbf{e}_{12}^2$, *where* $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$ *is the unit bivector of the plane containing* $\mathbf{v}$ *and* $\mathbf{w}$, *and show*

$$\mathbf{v}\wedge\mathbf{w} = \langle \mathbf{ve}_{12}\mathbf{w}\rangle_S\,\mathbf{e}_{12}\ .$$

Explain how this is a special case of the theorem immediately above.

In APS, the volume of the parallelepiped with sides $\mathbf{a},\mathbf{b},\mathbf{c}$, is the dual of the trivector

$$\begin{aligned}
\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} &= \langle\mathbf{abc}\rangle_3 = \langle\mathbf{abc}\rangle_{\Im} \\
&= \langle\langle\mathbf{ab}\rangle_2\,\mathbf{c}\rangle_{\Im} = \langle i\ {}^*\langle\mathbf{ab}\rangle_2\,\mathbf{c}\rangle_{\Im} = i\,\langle(\mathbf{a}\times\mathbf{b})\,\mathbf{c}\rangle_{\Re} \\
&= i\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}
\end{aligned}$$

where $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_T$ is the unit trivector of the algebra.

4.2 Reciprocal Basis

Except for their normalization, reciprocal basis vectors are duals to hyperplanes formed by wedging all but one of the basis vectors. The reciprocal basis is important when the basis is not orthogonal and not necessarily normalized, as in the study of crystalline solids. Thus, if we form a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ from three non-coplanar vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, in APS, the reciprocal vector to $\mathbf{a}_1$ is

$$\mathbf{a}^1 \equiv \frac{^*(\mathbf{a}_2 \wedge \mathbf{a}_3)}{^*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = \frac{\mathbf{a}_2 \wedge \mathbf{a}_3}{\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3} = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)},$$

where we noted that $^*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)$ is a real scalar, so that

$$\begin{aligned}
\mathbf{a}_1 \cdot \mathbf{a}^1 &= \frac{\mathbf{a}_1 \cdot {}^*(\mathbf{a}_2 \wedge \mathbf{a}_3)}{^*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = \frac{^*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)}{^*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = 1 \\
\mathbf{a}_2 \cdot \mathbf{a}^1 &= \frac{\mathbf{a}_2 \cdot {}^*(\mathbf{a}_2 \wedge \mathbf{a}_3)}{^*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = \frac{^*(\mathbf{a}_2 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)}{^*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = 0 = \mathbf{a}_3 \cdot \mathbf{a}^1 \ .
\end{aligned}$$

We can think of the reciprocal vectors as 1-forms, that is linear operators on vectors whose operation is defined by

$$\mathbf{a}^k(\mathbf{a}_j) = \mathbf{a}_j \cdot \mathbf{a}^k = \delta_j^k \ .$$

5 Paravectors and Relativity

The space of scalars, the space of vectors, and the space of bivectors, are all linear subspaces of the full 2^n-dimensional space of the algebra. Direct sums of the subspaces are also linear subspaces of the algebra. The most important is the direct sum of the scalar and the vector subspaces. It is an $(n+1)$-dimensional linear space known as *paravector space*. [In the algebra of physical space ($C\ell_3$), every element reduced to a complex paravector.]

Elements of paravector space have the form $p = p^0 + \mathbf{p} = \langle p \rangle_0 + \langle p \rangle_1$, and the algebra $C\ell_n$ also includes their exterior products:

$$\begin{aligned}
\text{paravector space} &= \langle C\ell_n \rangle_{1,0} \ , \ (n+1)\text{-dim} \\
\text{biparavector space} &= \langle C\ell_n \rangle_{2,1} \ , \ \binom{n+1}{2}\text{-dim} \\
\text{k-paravector space} &= \langle C\ell_n \rangle_{k,k-1} \ , \ \binom{n+1}{k}\text{-dim} \ .
\end{aligned}$$

In general, grade-0 paravectors are scalars in $\langle C\ell_n \rangle_0$, $(n+1)$-grade paravectors are volume elements (pseudoscalars) in $\langle C\ell_n \rangle_n$, and the linear space of k-grade multiparavectors is $\langle C\ell_n \rangle_{k,k-1} \equiv \langle C\ell_n \rangle_k \oplus \langle C\ell_n \rangle_{k-1}$, $k = 1, 2, \ldots, n$.

5.1 Conjugations

We need two conjugations (anti-automorphisms) and their combination. For any paravector $p = \langle p \rangle_{1,0}$,

$$\begin{aligned} \text{Clifford (bar) conjugation } \bar{p} &= p^0 - \mathbf{p},\ \overline{pq} = \bar{q}\bar{p} \\ \text{reversal (dagger) conjugation } p^\dagger &= p,\ (pq)^\dagger = q^\dagger p^\dagger \\ \text{grade automorphism } \bar{p}^\dagger &= \bar{p},\ (\overline{pq})^\dagger = \overline{(pq)^\dagger} = \bar{p}^\dagger \bar{q}^\dagger . \end{aligned}$$

The orthonormal basis vectors of a Euclidean space can all be represented by hermitian matrices, and reversal is then the same as hermitian conjugation. Recognizing this possibility, we adopt the dagger notation for reversal in $C\ell_n$.

The conjugations can be used to split elements in various ways:

$$\begin{aligned} p &= \frac{p+\bar{p}}{2} + \frac{p-\bar{p}}{2} = \langle p \rangle_S + \langle p \rangle_V = \text{scalarlike} + \text{vectorlike} \\ &= \frac{p+p^\dagger}{2} + \frac{p-p^\dagger}{2} = \langle p \rangle_\Re + \langle p \rangle_\Im = \text{real} + \text{imaginary} \\ &= \frac{p+\bar{p}^\dagger}{2} + \frac{p-\bar{p}^\dagger}{2} = \langle p \rangle_+ + \langle p \rangle_- = \text{even} + \text{odd}. \end{aligned}$$

These relations offer simple ways to isolate different vector and paravector grades. In particular, for $n = 3$, (here $\cdots$ stands for any expression)

$$\begin{aligned} \langle \cdots \rangle_S &= \langle \cdots \rangle_{0,3} \quad \langle \cdots \rangle_V = \langle \cdots \rangle_{1,2} \\ \langle \cdots \rangle_\Re &= \langle \cdots \rangle_{0,1} \quad \langle \cdots \rangle_\Im = \langle \cdots \rangle_{2,3} \\ \langle \cdots \rangle_+ &= \langle \cdots \rangle_{0,2} \quad \langle \cdots \rangle_- = \langle \cdots \rangle_{1,3} \ . \end{aligned}$$

Exercise 43 *Verify that the splits can be combined to extract individual vector grades as follows:*

$$\begin{aligned} \langle \cdots \rangle_0 &= \langle \cdots \rangle_{\Re S} = \langle \cdots \rangle_{\Re +} = \langle \cdots \rangle_{S+} \\ \langle \cdots \rangle_1 &= \langle \cdots \rangle_{\Re V} \\ \langle \cdots \rangle_2 &= \langle \cdots \rangle_{\Im V} \\ \langle \cdots \rangle_3 &= \langle \cdots \rangle_{\Im S} \ . \end{aligned}$$

Example 3 *Let* $\mathbf{B}$ *be any bivector.*
$\mathbf{B}$ *is even, imaginary, and vectorlike.*
$\mathbf{B}^2$ *is even, real, and scalarlike.*
Any analytic function $f(\mathbf{B})$ *is even and* $f(-\mathbf{B}) = f(\bar{\mathbf{B}}) = f(\mathbf{B}^\dagger)$
Spatial rotors $R(\mathbf{B}) = \exp(\mathbf{B}/2)$ *are even and unitary:* $R^\dagger(\mathbf{B}) = R^{-1}(\mathbf{B}) = R(-\mathbf{B})$

We can use either grade numbers or conjugation symmetries to split an element into parts. The grade numbers emphasize the algebraic structure whereas the conjugation symmetries indicate an operational procedure to compute the part.

5.2 Paravector basis and metric

If $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is an orthonormal basis of the original Euclidean space, so that

$$\langle \mathbf{e}_j \mathbf{e}_k \rangle_0 = \frac{1}{2}(\mathbf{e}_j \mathbf{e}_k + \mathbf{e}_k \mathbf{e}_j) = \delta_{jk} ,$$

the proper basis of paravector space is $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, where we identify $\mathbf{e}_0 \equiv 1$ for convenience in expanding paravectors $p = p^\mu \mathbf{e}_\mu,\ \mu = 0, 1, \cdots, n$ in the basis. The metric of paravector space is determined by the quadratic form. We need a product of a paravector p with itself or a conjugate that is scalar valued. It is easy to see that p^2 generally has vector parts, but $p\bar{p} = \langle p\bar{p} \rangle_0 = \bar{p}p$ is a scalar. Therefore it is adopted as the *quadratic form* ("square length"):

$$Q(p) = p\bar{p}.$$

By "polarization" $p \to p + q$ we find the inner product

$$\begin{aligned} \langle p, q \rangle &= \langle p\bar{q} \rangle_0 = \frac{1}{2}(p\bar{q} + q\bar{p}) \\ &= p^\mu q^\nu \langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_0 \equiv p^\mu q^\nu \eta_{\mu\nu} . \end{aligned}$$

Exercise 44 *Show that the inner product* $\langle p\bar{q} \rangle_0 = \frac{1}{2}[Q(p+q) - Q(p) - Q(q)]$.

Exercise 45 *Find the values of* $\eta_{\mu\nu} = \langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_0$.

We recognize the matrix

$$(\eta_{\mu\nu}) = \mathrm{diag}\,(1, -1, -1, \cdots, -1)$$

as the natural metric of the paravector space. It has the form of the *Minkowski metric.* If $\langle p\bar{q}\rangle_0 = 0$, then the paravectors p and q are *orthogonal.* For any element x, $x\bar{x} = \overline{x\bar{x}} = \langle x\bar{x}\rangle_S$. In the standard matrix rep of $C\ell_3$,

$$x\bar{x} \simeq \det \underline{x} \ .$$

If $x\bar{x} = 1$, x is *unimodular.*

The inverse of an element x can be written

$$x^{-1} = \frac{\bar{x}}{x\bar{x}},$$

but this doesn't exist if $x\bar{x} = 0$. The existence of nonzero elements of zero length means that geometric algebra, unlike the algebras of reals, complexes, and quaternions, is generally *not* a division algebra. This may seem an annoyance at first, but it is the basis for powerful projector techniques, as we demonstrate below.

Exercise 46 *Demonstrate that the paravector* $1 + \mathbf{e}_1$ *has no inverse and is orthogonal to itself.*

5.3 Spacetime as paravector space

The paravectors of physical space provide a covariant model of spacetime. Take $n = 3$ and use SI units. Spacetime vectors are represented by paravectors whose frame-dependent split into scalar and vector parts reflects the observer's ability to distinguish time and space components. Thus, the dimensionless proper velocity is

$$u = \frac{dx}{d\tau} = \gamma\left(1 + \mathbf{v}/c\right) = u^{\mu}\mathbf{e}_{\mu} \ ,$$

where $\mathbf{v}$ is the usual coordinate velocity vector, we use the summation convention for repeated indices, and for timelike displacements dx, the dimensionless proper time is the Lorentz scalar related by

$$d\tau^2 = dx d\bar{x} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}.$$

Similarly,

$$\begin{aligned} p &= mcu = E/c + \mathbf{p} : \text{ paramomentum} \\ j &= j^{\mu}\mathbf{e}_{\mu} = \rho c + \mathbf{j} : \text{current density} \\ A &= A^{\mu}\mathbf{e}_{\mu} = \phi/c + \mathbf{A} : \text{paravector potential} \\ \partial &= \partial_{\mu}\mathbf{e}_{\nu}\eta^{\mu\nu} = c^{-1}\partial_t - \nabla : \text{gradient operator.} \end{aligned}$$

Biparavectors represent *oriented planes* in spacetime, for example

$$\begin{aligned} \mathbf{F} &= c\left\langle \partial \bar{A} \right\rangle_{1,2} = \frac{1}{2} F^{\mu\nu} \left\langle \mathbf{e}_{\mu}\bar{\mathbf{e}}_{\nu} \right\rangle_{1,2} = \mathbf{E} + ic\mathbf{B} : \text{electromagnetic field} \\ \left\langle \mathbf{e}_{\mu}\bar{\mathbf{e}}_{\nu} \right\rangle_{1,2} &= -\left\langle \mathbf{e}_{\nu}\bar{\mathbf{e}}_{\mu} \right\rangle_{1,2} : \text{basis biparavectors} \rightarrow \text{ Lorentz rotations.} \end{aligned}$$

Thus, the electromagnetic field $\mathbf{F}$ is associated with the usual tensor components $F^{\mu\nu}$, but we don't need any tensor to express it. It is a covariant plane in spacetime which for the observer splits naturally into frame-dependent parts

$$\mathbf{F} = \mathbf{F}\mathbf{e}_0\mathbf{e}_0 = \left\langle \mathbf{F}\mathbf{e}_0 \right\rangle_{1,0} \mathbf{e}_0 + \left\langle \mathbf{F}\mathbf{e}_0 \right\rangle_{3,2} \mathbf{e}_0$$

whose timelike component is the electric field

$$\mathbf{E} = \left\langle \mathbf{F}\mathbf{e}_0 \right\rangle_{1,0} \mathbf{e}_0 = \left\langle \mathbf{F} \right\rangle_1$$

and whose spacelike part gives the magnetic field

$$ic\mathbf{B} = \left\langle \mathbf{F}\mathbf{e}_0 \right\rangle_{3,2} \mathbf{e}_0 = \left\langle \mathbf{F} \right\rangle_2 \ .$$

In fact the usual magnetic field (times c) $c\mathbf{B}$ is the vector dual to the spatial plane $\left\langle \mathbf{F} \right\rangle_2$ or equivalently to the spacetime hypersurface $\left\langle \mathbf{F}\mathbf{e}_0 \right\rangle_{3,2}$.

We distinguish *simple* fields, which are single spacetime planes from *compound* ones, which occupy two orthogonal (and hence commuting) planes. How do we tell them apart? All we need to do is take the square of the field. The square of any simple field is a real scalar, whereas the square of a compound field contains a volume element, that is a pseudoscalar, represented in the paravector model as an imaginary scalar.

The biparavector $\langle p\bar{q}\rangle_{2,1} = \frac{1}{2}(p\bar{q} - q\bar{p})$ represents a spacetime plane containing paravectors p and q. Its square

$$\begin{aligned} \langle p\bar{q}\rangle_{2,1}^2 &= \frac{1}{4}(p\bar{q} - q\bar{p})^2 = \frac{1}{4}(p\bar{q} + q\bar{p})^2 - \frac{1}{2}(p\bar{q}q\bar{p} + q\bar{p}p\bar{q}) \\ &= \langle p\bar{q}\rangle_0^2 - p\bar{p}q\bar{q} \end{aligned}$$

is seen to be a real scalar. The square of $\mathbf{F} = \mathbf{E} + ic\mathbf{B}$ is

$$\mathbf{F}^2 = \mathbf{E}^2 - c^2\mathbf{B}^2 + 2ic\mathbf{E}\cdot\mathbf{B}$$

which means that $\mathbf{F}$ is evidently simple if and only if $\mathbf{E}\cdot\mathbf{B} = 0$.

A null field has $\mathbf{F}^2 = 0$ and can be written $\mathbf{F} = \left(1 + \hat{\mathbf{k}}\right)\mathbf{E}$, where $\hat{\mathbf{k}}\mathbf{E} = ic\mathbf{B}$.

Exercise 47 *Show that any null field* $\mathbf{F} = \left(1 + \hat{\mathbf{k}}\right)\mathbf{E}$ *obeys* $\hat{\mathbf{k}}\mathbf{F} = \mathbf{F} = -\mathbf{F}\hat{\mathbf{k}}$.

5.4 Lorentz transformations

Physical (restricted) Lorentz transformations are rotations in paravector space. They take the form of spin transformations

$$\begin{aligned} p &\rightarrow LpL^\dagger, \text{ odd multiparavector grade} \\ \mathbf{F} &\rightarrow L\mathbf{F}\bar{L}\ , \text{ even multiparavector grade,} \end{aligned}$$

where the Lorentz rotors L are unimodular ($L\bar{L} = 1$) and have the form

$$\begin{aligned} L &= \exp(\mathbf{W}/2) \in SL(2,\mathbb{C}) \\ \mathbf{W} &= \frac{1}{2}W^{\mu\nu}\langle \mathbf{e}_\mu\bar{\mathbf{e}}_\nu\rangle_{1,2}\ . \end{aligned}$$

Every L can be factored into a boost $B = B^\dagger$ (a real factor) and a spatial rotation (a unitary factor) $R = \bar{R}^\dagger : L = BR$.

For any paravectors p, q, the square lengths $p\bar{p}, q\bar{q}$ are Lorentz invariant, as is the scalar product $\langle p\bar{q}\rangle_S$. A paravector p, can be timelike ($p\bar{p} > 0$), spacelike ($p\bar{p} < 0$), or lightlike (null, $p\bar{p} = 0$) and this is an invariant property. Null paravectors are orthogonal to themselves. Similarly $\mathbf{F}^2$ is also Lorentz invariant and simple fields can be classified as predominantly electric ($\mathbf{F}^2 > 0$), predominantly magnetic ($\mathbf{F}^2 < 0$), or null ($\mathbf{F}^2 = 0$).

The *position coordinate* x of a particle *at rest* changes only by its time, the proper time τ :

$$dx_{\rm rest} = d\tau$$

Let's transform this to the lab, in which the particle moves with proper velocity $u = dx/d\tau$:

$$\begin{aligned} dx &= L dx_{\rm rest} L^\dagger = LL^\dagger d\tau = u d\tau \\ &= c dt + d\mathbf{x} = dt\,(c + \mathbf{v}) \end{aligned}$$

Thus,

$$LL^\dagger = B^2 = u = \frac{dt}{d\tau}\,(c + \mathbf{v})\,.$$

Now $LL^\dagger = L\mathbf{e}_0 L^\dagger = u$ is just the Lorentz rotation of the unit basis paravector in the time direction, and since its length is invariant,

$$u\ \bar{u} = 1 = \gamma^2 \left(1 - \mathbf{v}^2/c^2\right)$$

where $\gamma = c dt/d\tau$ is the *time-dilation factor.*

Example 4 *Consider the transformation of a paravector* $p = p^\mu \mathbf{e}_\mu$ *in a system that is boosted from rest to a velocity* $\mathbf{v} = v\mathbf{e}_3$:

$$p \rightarrow LpL^\dagger = BpB = p^\mu \mathbf{u}_\mu$$

where $B = \exp\left(w\mathbf{e}_3\bar{\mathbf{e}}_0/2\right) = u^{1/2}$ *represents a rotation in the* $\mathbf{e}_3\bar{\mathbf{e}}_0$ *paravector plane and* $\mathbf{u}_\mu = B\mathbf{e}_\mu B$ *is the boosted proper basis paravector. Evidently*

$$\begin{aligned} \mathbf{u}_0 &= B\mathbf{e}_0 B = B^2\mathbf{e}_0 = u\mathbf{e}_0 = \gamma\left(\mathbf{e}_0 + \frac{v}{c}\mathbf{e}_3\right) \\ \mathbf{u}_1 &= \mathbf{e}_1\ ,\ \mathbf{u}_2 = \mathbf{e}_2 \\ \mathbf{u}_3 &= B\mathbf{e}_3 B = u\mathbf{e}_3 = \gamma\left(\mathbf{e}_3 + \frac{v}{c}\mathbf{e}_0\right) \end{aligned}$$

with $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$.

As in the case of spatial rotations, if we put $LpL^\dagger = p' = p'^\nu \mathbf{e}_\nu$, *we can easily find*

$$p'^\nu = p^\mu \left\langle \mathbf{u}_\mu \bar{\mathbf{e}}^\nu \right\rangle_S$$

and thus the usual 4×4 *matrix relating the components of* p *before and after the boost, but we don't really need it. The relations for* $\mathbf{u}_\mu$ *are useful for drawing spacetime diagrams. Thus, if* $v = 0.6\ c$, *then* $\gamma = 1.25$ *and*

$$\begin{aligned} \mathbf{u}_0 &= \left(5\mathbf{e}_0 + 3\mathbf{e}_3\right)/4 \\ \mathbf{u}_3 &= \left(5\mathbf{e}_3 + 3\mathbf{e}_0\right)/4\ . \end{aligned}$$

We can take this further and look at planes in spacetime, as shown in the figure.

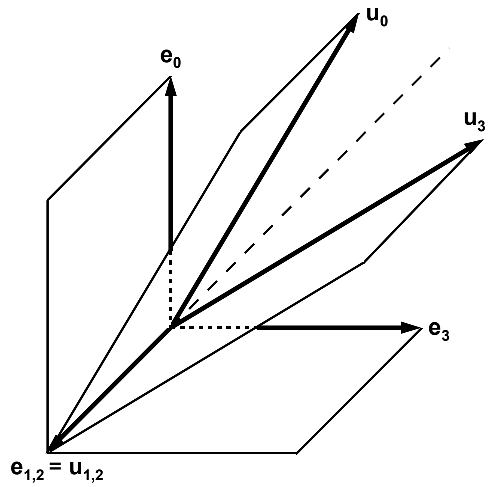

Figure 8: Spacetime diagram showing the boost to $\mathbf{v} = 0.6\, c\mathbf{e}_3$.

Exercise 48 *Show that the biparavectors* $\mathbf{e}_3\bar{\mathbf{e}}_0$ *and* $\mathbf{e}_1\bar{\mathbf{e}}_2$ *are invariant under any boost* B *along* $\mathbf{e}_3$*.*

Exercise 49 *Let system* B *have proper velocity* u_{AB} *with respect to* A*, and let system* C *have proper velocity* u_{BC} *as seen by an observer in* B*. Show that the proper velocity of* C *as viewed by* A *is*

$$u_{AC} = u_{AB}^{1/2} u_{BC} u_{AB}^{1/2}$$

and that this reduces to the product $u_{AC} = u_{AB}u_{BC}$ *when the spatial velocities are collinear. Writing each proper velocity in the form* $u = \gamma\,(1 + \mathbf{v})$ *, show that in the collinear case*

$$\mathbf{v}_{AC} = \frac{\langle u_{AC}\rangle_V}{\langle u_{AC}\rangle_S} = \frac{\mathbf{v}_{AB} + \mathbf{v}_{BC}}{1 + \mathbf{v}_{AB}\cdot\mathbf{v}_{BC}}.$$

Example 5 *Consider a boost of the photon wave paravector*

$$k = \frac{\omega}{c}\left(1+\hat{\mathbf{k}}\right) \rightarrow k' = BkB = u\left(\frac{\omega}{c}+\mathbf{k}^{\|}\right)+\mathbf{k}^{\perp}$$

with $\mathbf{k}^{\|} = \mathbf{k}\cdot\hat{\mathbf{v}}\ \hat{\mathbf{v}} = \mathbf{k}-\mathbf{k}^{\perp}$ *and* $u = \gamma\left(1+\mathbf{v}/c\right)$*. This describes what happens to the photon momentum when the light source is boosted. Evidently* $\mathbf{k}^{\perp}$ *is unchanged, but there is a Doppler shift and a change in* $\mathbf{k}^{\|}$ *:*

$$\begin{aligned} \omega' &= \left\langle u\left(\omega + c\mathbf{k}^{\|}\right)\right\rangle_S = \gamma\omega\left(1+\hat{\mathbf{k}}\cdot\mathbf{v}/c\right) \\ \mathbf{k}'\cdot\hat{\mathbf{v}} &= \left\langle u\hat{\mathbf{v}}\left(\frac{\omega}{c}+\mathbf{k}^{\|}\right)\right\rangle_S = \gamma\frac{\omega}{c}\left(\frac{v}{c}+\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}\right) = \frac{\omega'}{c}\cos\theta'. \end{aligned}$$

Thus, the photons are thrown forward:

$$\cos\theta' = \frac{v + c\cos\theta}{c+v\cos\theta}\ . \tag{9}$$

This is the "headlight" effect:

Exercise 50 *Solve Eq. (9) for* $\cos\theta$ *and show that the result is the same as in Eq. (9) except that* v *is replaced by* $-v$ *and* θ *and* θ' *are interchanged.*

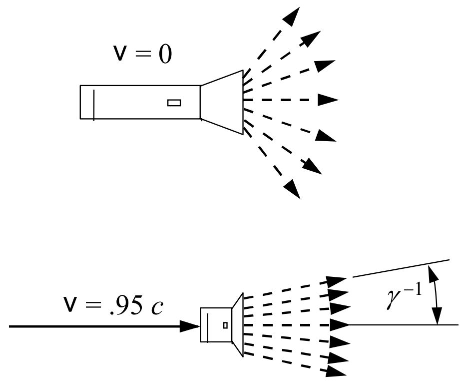

Figure 9: Headlight effect in boosted light source.

Exercise 51 *Show that at high velocities, the radiation from the boosted source is concentrated in the cone of angle γ^{-1} about the forward direction.*

Remark 5 *D. Hestenes' treatment of relativity in his book* New Foundations for Classical Mechanics, *2/e (Kluwer Academic, 1999), c.9, is equivalent to the paravector model of relativity presented here. In his earlier book* Space-Time Algebra *(Gordon and Breach, 1966), he instead uses the space-time algebra (STA), that is the real algebra $C\ell_{1,3}$, whose even part is isomorphic to APS.*

6 Conclusions

This workbook has given a bare introduction to the application of GA, and in particular the APS dialect of GA, to problems in physics. A number of problems have been worked out in detail with the hope of providing a real sense of its application in the contemporary physics curriculum. I hope the following points have been demonstrated:

- The basis of GA is very simple. It is an algebra of vectors that naturally unifies vector products with complex analysis and extends these techniques to higher dimensions.

- GA replaces much of the matrix and tensor methods in the physics curriculum with an algebra that emphasizes geometric concepts.

- In GA, and in particular in the APS dialect, relativistic treatments are almost as easy to formulate, calculate, and interpret as Galilean/Newtonian ones. The conceptual and computational importance of covariant relativistic symmetries makes it advantageous to use relativity at early stages in courses on mechanics, electromagnetism, and quantum theory. GA makes this possible.

- GA makes the classical/quantum interface more transparent. Rotors are amplitudes like quantum wave functions, and spinors and projectors provide powerful tools in the classical realm. The GA approach integrates quantum theory smoothly into the rest of the physics curriculum.

- GA is beautiful, powerful, and fun! Its use can make the undergraduate physics curriculum more efficient and more stimulating.

There is a wealth of information on GA and STA in books and journal publications by Hestenes and co-workers. See *http://modelingnts.la.asu.edu* for a listing and downloadable copies of many articles. More information on the APS approach can be found at the website *http://www.uwindsor.ca/baylis-research/* and in the following publications:

Introduction and general:

W. E. Baylis, J. Huschilt, and J. Wei, *Am. J. Phys.* **60**, 788 (1992).

W. E. Baylis and G. Jones, *J. Phys. A* **22**, 1; 17 (1989).

W. E. Baylis, *Clifford (Geometric) Algebras with Applications in Physics, Mathematics, and Engineering*, Birkhäuser Boston, 1996.

Electrodynamics:

W. E. Baylis, *Electrodynamics, A Modern Geometric Approach,* Birkhäuser Boston, 1999. Second printing with minor corrections 2002.

W. E. Baylis and Y. Yao, *Phys. Rev. A* **60,** 785 (1999).

Dirac equation:

W. E. Baylis, *Phys. Rev. A***45**, 4293 (1992).

W. E. Baylis, *Adv. Appl. Clifford Algebras* **7(S)** 197 (1997).

Standard Model

G. Trayling and W. E. Baylis, *J. Phys. A* **34**, 3309 (2001).

Acknowledgement

Support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.